

# A Weighted Large Sieve Inequality

Anish Ray

In this article, we obtain a large-sieve type estimate by introducing a weight function  $\omega$  in the right-hand side as defined below.

The large sieve inequality in its linear algebraic version states that:

**Theorem 1.** *Let  $(V, \langle \cdot, \cdot \rangle)$  be an inner product space, where the elements of  $V$  are  $\xi, \varphi_1, \dots, \varphi_R$ ,  $R \geq 1$ . Then, we have*

$$\sum_{i=1}^R |\langle \xi, \varphi_i \rangle|^2 \leq \|\xi\|^2 \max_i \sum_j |\langle \varphi_i, \varphi_j \rangle|. \quad (1)$$

Let  $M \geq 1$  be an integer and we define the weight function  $\omega(n)$  as follows:

$$\begin{aligned} \omega(N-M+k) &:= \frac{13}{8} \frac{k}{M}, \quad 0 \leq k \leq M \text{ for } N-M \leq n = N-M+k \leq N-1, \\ \omega(n) &:= -\frac{5}{4} \frac{n}{N} + \frac{23}{8} \text{ for } N \leq n \leq 2N, \\ \omega(2N+M-k) &:= \frac{3}{8} \frac{k}{M}, \quad 0 \leq k \leq M \text{ for } 2N \leq n = 2N+M-k \leq 2N+M, \\ \omega(n) &:= 0, \text{ otherwise.} \end{aligned}$$

We define the inner product to be  $\langle u, v \rangle = \sum_{n=1}^N \omega(n) u_n \overline{v_n}$ , the entries in  $\xi$  are  $(a_n)$  and

for integers  $a, q \geq 1$  the entries in  $\{\varphi_i\}$  are an enumeration of  $\left\{ e\left(\frac{a}{q}n\right) \mid \gcd(a, q) = 1, q \leq Q \right\}$  for an integer  $Q \geq 1$ , where  $e(x) = e^{2\pi i x}$ .

**Theorem 2.** *Let  $N, M \geq 1$  be integers and  $\omega$  be defined as above. Let  $(a_n)$  be a sequence of complex numbers. Then, using Theorem 1 with  $\xi$  and  $\{\varphi_i\}$  as defined above, we obtain*

$$\sum_{q \leq Q} \sum_{a=1}^q \left| \sum_{n=N-M}^{2N+M} \omega(n) a_n e\left(\frac{a}{q}n\right) \right|^2 \leq \left( N + \frac{2}{\sqrt{3}} Q^2 \right) \left( \sum_{n=N-M}^{2N+M} \omega(n) |a_n|^2 \right). \quad (2)$$

*Proof.* Using Theorem 1, we get

$$\sum_{q \leq Q} \sum_{a=1}^q \left| \sum_{n=N-M}^{2N+M} \omega(n) a_n e\left(\frac{a}{q}n\right) \right|^2 \leq \left( \sum_{n=N-M}^{2N+M} \omega(n) |a_n|^2 \right) \max_i \sum_j \underbrace{\left| \sum_{n=N-M}^{2N+M} \omega(n) e\left(\left(\frac{a_i}{q_i} - \frac{a_j}{q_j}\right)n\right) \right|}_{S_j}. \quad (3)$$

For the sake of simplicity, we write  $\alpha = \frac{a_i}{q_i}$ ,  $\beta = \frac{a_j}{q_j}$ , and let  $\|\alpha - \beta\| \geq \frac{1}{Q^2}$ . Let us first obtain an estimate of  $S_j$ .

$$\begin{aligned} S_j = & \left| \underbrace{\frac{13}{8M} \sum_{k=1}^{M-1} k e((\alpha - \beta)(N - M + k))}_{S_1} - \frac{5}{4N} \underbrace{\sum_{n=N}^{2N} n e((\alpha - \beta)n)}_{S_2} + \frac{23}{8} \underbrace{\sum_{n=N}^{2N} e((\alpha - \beta)n)}_{S_3} \right. \\ & \left. + \frac{3}{8M} \underbrace{\sum_{k=1}^{M-1} k e((\alpha - \beta)(2N + M - k))}_{S_4} \right| \end{aligned} \quad (4)$$

We want to evaluate the sums  $S_1, S_2, S_3$  and  $S_4$  one by one. To this end, we note that

$$\begin{aligned} \sum_{k=1}^{M-1} k e^{(N-M+k)x} &= e^{(N-M)x} \sum_{k=1}^{M-1} k e^{kx} = e^{(N-M)x} \frac{d}{dx} \left( \sum_{k=1}^{M-1} e^{kx} \right) = e^{(N-M)x} \frac{d}{dx} \left( \frac{e^{Mx} - e^x}{e^x - 1} \right) \\ &= e^{(N-M)x} \left[ \frac{(M e^{Mx} - e^x)(e^x - 1) - e^x (e^{Mx} - e^x)}{(e^x - 1)^2} \right] = e^{(N-M)x} \left[ \frac{(M-1)e^{(M+1)x} - M e^{Mx} + e^x}{(e^x - 1)^2} \right] \\ &= \frac{(M-1)e^{(N+1)x} - M e^{Nx} + e^{(N-M+1)x}}{(e^x - 1)^2} \end{aligned}$$

Then, for  $x = 2\pi i(\alpha - \beta)$ , we obtain

$$S_1 = \frac{1}{(e(\alpha - \beta) - 1)^2} \left[ (M-1)e((\alpha - \beta)(N+1)) - M e((\alpha - \beta)N) + e((\alpha - \beta)(N-M+1)) \right].$$

Next, we have

$$\begin{aligned} \sum_{n=N}^{2N} n e^{nx} &= \frac{d}{dx} \left( \sum_{n=N}^{2N} e^{nx} \right) = \frac{d}{dx} \left( \frac{e^{(2N+1)x} - e^{Nx}}{e^x - 1} \right) \\ &= \left[ \frac{((2N+1)e^{(2N+1)x} - N e^{Nx})(e^x - 1) - e^x (e^{(2N+1)x} - e^{Nx})}{(e^x - 1)^2} \right] \\ &= \frac{2N e^{(2N+2)x} - (2N+1)e^{(2N+1)x} - (N-1)e^{(N+1)x} + N e^{Nx}}{(e^x - 1)^2}. \end{aligned}$$

Then, for  $x = 2\pi i(\alpha - \beta)$ , we obtain

$$S_2 = \frac{1}{(e(\alpha - \beta) - 1)^2} \left[ 2N e((\alpha - \beta)(2N+2)) - (N-1)e((\alpha - \beta)(N+1)) - (2N+1)e((\alpha - \beta)(2N+1)) + e((\alpha - \beta)N) \right].$$

Next, we obtain

$$\sum_{n=N}^{2N} e^{nx} = \frac{e^{(2N+1)x} - e^{Nx}}{e^x - 1} = \frac{(e^{(2N+1)x} - e^{Nx})(e^x - 1)}{(e^x - 1)^2} = \frac{e^{(2N+2)x} - e^{(2N+1)x} - e^{(N+1)x} + e^{Nx}}{(e^x - 1)^2}.$$

Then, for  $x = 2\pi i(\alpha - \beta)$ , we obtain

$$S_3 = \frac{1}{(e(\alpha - \beta) - 1)^2} \left[ e(\alpha - \beta)(2N+2) - e((\alpha - \beta)(2N+1)) - e((\alpha - \beta)(N+1)) + e((\alpha - \beta)N) \right].$$

Lastly, we have

$$\begin{aligned} \sum_{k=1}^{M-1} k e^{(2N+M-k)x} &= -e^{(2N+M)x} \sum_{k=1}^{M-1} (-k) e^{-kx} = -e^{(2N+M)x} \frac{d}{dx} \left( \sum_{k=1}^{M-1} e^{-kx} \right) = -e^{(2N+M)x} \frac{d}{dx} \left( \frac{e^{-Mx} - e^{-x}}{e^{-x} - 1} \right) \\ &= -e^{(2N+M)x} \left[ \frac{(-M e^{Mx} + e^{-x})(e^{-x} - 1) + e^{-x}(e^{-Mx} - e^{-x})}{(e^{-x} - 1)^2} \right] = e^{(2N+M)x} \left[ \frac{(M-1)e^{-(M+1)x} - M e^{-Mx} + e^{-x}}{(e^{-x} - 1)^2} \right] \\ &= \frac{(M-1)e^{(2N-1)x} - M e^{2Nx} + e^{(2N+M-1)x}}{(e^{-x} - 1)^2} = \frac{(M-1)e^{(2N+1)x} - M e^{(2N+2)x} + e^{(2N+M+1)x}}{(e^x - 1)^2} \end{aligned}$$

Then, for  $x = 2\pi i(\alpha - \beta)$ , we obtain

$$S_4 = \frac{1}{(e(\alpha - \beta) - 1)^2} \left[ (M-1)e((\alpha - \beta)(2N+1)) - M e((\alpha - \beta)(2N+2)) + e((\alpha - \beta)(2N+M+1)) \right].$$

Then, we obtain

$$\begin{aligned} S_j &= \left| \frac{13}{8(e(\alpha - \beta) - 1)^2} \left[ \left(1 - \frac{1}{M}\right) e((\alpha - \beta)(N+1)) - e((\alpha - \beta)N) + \frac{1}{M} e((\alpha - \beta)(N-M+1)) \right] \right. \\ &\quad - \frac{5}{4(e(\alpha - \beta) - 1)^2} \left[ 2e((\alpha - \beta)(2N+2)) - \left(1 - \frac{1}{N}\right) e((\alpha - \beta)(N+1)) - \left(2 + \frac{1}{N}\right) e((\alpha - \beta)(2N+1)) + e((\alpha - \beta)N) \right] \\ &\quad \left. + \frac{23}{8(e(\alpha - \beta) - 1)^2} \left[ e(\alpha - \beta)(2N+2) - e((\alpha - \beta)(2N+1)) - e((\alpha - \beta)(N+1)) + e((\alpha - \beta)N) \right] \right. \\ &\quad \left. + \frac{3}{8(e(\alpha - \beta) - 1)^2} \left[ \left(1 - \frac{1}{M}\right) e((\alpha - \beta)(2N+1)) - e((\alpha - \beta)(2N+2)) + \frac{1}{M} e((\alpha - \beta)(2N+M+1)) \right] \right| \\ &= \left| \frac{1}{e(\alpha - \beta) - 1)^2} \left[ \left( \frac{5}{4N} - \frac{3}{8M} \right) e((\alpha - \beta)(2N+1)) - \left( \frac{5}{4N} + \frac{13}{8M} \right) e((\alpha - \beta)(N+1)) + \frac{3}{8M} e((\alpha - \beta)(2N+M+1)) \right. \right. \\ &\quad \left. \left. + \frac{13}{8M} e((\alpha - \beta)(N-M+1)) \right] \right| \end{aligned}$$

For a fix  $i \in \{1, \dots, R\}$ ,  $\alpha = \frac{a_i}{q_i}$  is fixed. Then for  $j \neq i$ ,  $\alpha \neq \beta$ , and we have

$$S_j \leq \frac{1}{4 \sin^2(\pi(\alpha - \beta))} \left( \left| \frac{5}{4N} - \frac{3}{8M} \right| + \frac{5}{4N} + \frac{29}{8M} \right) \leq \frac{1}{\sin^2(\pi(\alpha - \beta))} \left( \frac{5}{8N} + \frac{1}{M} \right). \quad (5)$$

When  $\alpha = \beta$ , from (4) we obtain,

$$S_j = \frac{13}{16}(M-1) - \frac{5}{8}(3N+3) + \frac{23}{8}(N+1) + \frac{3}{16}(M-1) = N+M. \quad (6)$$

Then we sum up  $S_j$  over  $j$ , and using the inequality  $\frac{1}{\sin^2(\pi(\alpha-\beta))} \leq \frac{1}{\pi^2|\alpha-\beta|^2} + 1$ , we proceed in the steps of Montgomery ([1], page no. 18). So, it follows from (5) and (6), that

$$\begin{aligned} \sum_j S_j &\leq N+M + \left(\frac{5}{8N} + \frac{1}{M}\right) \left(R + \frac{2}{\pi^2} \sum_{j=1}^{\frac{R+1}{2}} \left\| \frac{Q^2}{j} \right\|^2\right) \\ &< N+M + \left(\frac{5}{8N} + \frac{1}{M}\right) \left(R + \frac{Q^4}{3}\right). \end{aligned} \quad (7)$$

Moreover, one obtains

$$R = \sum_{q \leq Q} \phi(q) \sim \frac{3}{\pi^2} Q^2, \quad (8)$$

where  $\phi$  is the Euler's Totient function. So, the left-hand side of the last inequaity in (7), can be rewritten as  $N+M + \left(\frac{5}{8N} + \frac{1}{M}\right) \left(\frac{3}{\pi^2} Q^2 + \frac{Q^4}{3}\right)$ . If we consider this as a function of  $M$ , we obtain that the minima occurs at  $M = \left\lfloor \sqrt{\frac{3}{\pi^2} Q^2 + \frac{Q^4}{3}} \right\rfloor$ . Then, using

$\frac{Q^2}{\sqrt{3}} \leq \left\lfloor \sqrt{\frac{3}{\pi^2} Q^2 + \frac{Q^4}{3}} \right\rfloor \leq \frac{Q^2}{\sqrt{3}} + \frac{3\sqrt{3}}{2\pi^2}$ , we obtain

$$\begin{aligned} M + \frac{3Q^2}{\pi^2 M} + \frac{Q^4}{3M} &< \frac{Q^2}{\sqrt{3}} + \frac{3\sqrt{3}}{2\pi^2} + \frac{3\sqrt{3}}{\pi^2} + \frac{Q^2}{\sqrt{3}} \\ &= \frac{2Q^2}{\sqrt{3}} + \frac{9\sqrt{3}}{2\pi^2} \end{aligned} \quad (9)$$

Hence, for  $Q$  and  $N$  large enough, we obtain (2).  $\square$

## References

- [1] Montgomery, Hugh L. Topics in multiplicative number theory. Vol. 227. Springer, 2006.