Homogenization of Convex Integral Functionals Under Weakened Growth Conditions

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Introduction

We consider a family of functionals of the type

$$I^{\varepsilon}(u) = \int_{\Omega} W\left(\frac{x}{\varepsilon}, \nabla u(x)\right) \mathrm{d}x, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^{\mathrm{N}}$ is bounded, $u : \Omega \to \mathbb{R}^m$ is a vector valued function and $W : \mathbb{R}^{\mathrm{N}} \times \mathbb{R}^{m\mathrm{N}} \to \mathbb{R}$ is assumed to be measurable and periodic on $Y = [0,1]^{\mathrm{N}}$ with respect to its first variable. The deformation gradient of u is denoted by ∇u and $\varepsilon > 0$ is a (small) parameter. Our goal is to show that as $\varepsilon \to 0$ then I^{ε} converges to simpler **"homogenized"** functional

$$I(u) = \int_{\Omega} W_{hom}(\nabla u(x)) \mathrm{d}x, \tag{1.2}$$

in a sense to be explained later. Furthermore we wish to give a representation formula for W_{hom}. This problem has been solved by MARCELLINI in 1978 for scalar u and W convex and of polynomial growth with respect to the deformation gradient. We want to prove MARCELLINI's conclusion under weakened growth conditions.

Some defintions

Definition 1

Let $\{I^\varepsilon\}_{\varepsilon>0}$ be a family of functionals on $W^{1,p}(\Omega)$ $(1 < p < \infty)$. We say that $\{I^\varepsilon\}_{\varepsilon>0}$ is Γ-convergent to a functional I (with respect to the weak topology of $W^{1,p}(\Omega)$ if
(i) for every sequence $\{u^\varepsilon\}_{\varepsilon>0}$ with $u^\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega)$ we have

$$\liminf_{\varepsilon\to 0} I^\varepsilon(u^\varepsilon) \geq I(u) \tag{1.3}$$

(ii) for every $u \in W^{1,p}(\Omega)$ there is a sequence $\{u^\varepsilon\}_{\varepsilon>0}$ with $u^\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega)$ and

$$\lim_{\varepsilon\to 0} I^\varepsilon(u^\varepsilon) = I(u). \tag{1.4}$$

The result on the convergence of minimizers is contained in the following lemma.

Lemma 1.1

Assume that $\{I^\varepsilon\}_{\varepsilon>0}$ *is* Γ*-convergent to* I *and* g *is a continuous linear functional on* $W^{1,p}(\Omega)$. *Let* $u^\varepsilon \in W^{1,p}(\Omega)$ *be such that*

$$I^\varepsilon(u^\varepsilon) + g(u^\varepsilon) < \inf\{I^\varepsilon(u) + g(u) \mid u \in W^{1,p}(\Omega)\} + \varepsilon$$

assume furthermore that $\{u^\varepsilon\}_\varepsilon > 0$ *is weakly compact in* $W^{1,p}(\Omega)$. *Let* u^{ε_n} *be any weakly convergent sequence, say* $u^{\varepsilon_n} \rightharpoonup u$ *in* $W^{1,p}(\Omega)$ *as* $\varepsilon_n \to 0$. *Then*
(i) $I(u) + g(u) \leq I(v) + g(v) \, \forall v \in W^{1,p}(\Omega)$,
(ii) $\min\{I(u) + g(u) \mid u \in W^{1,p}(\Omega)\} = \liminf_{\varepsilon\to 0}\{I^\varepsilon(u) + g(u) \mid u \in W^{1,p}(\Omega)\}$. (1.5)

MARCELLINI showed that when W is convex and has polynomial growth with respect to the the deformation gradient, the Γ- limit is given by

$$I(u) = \int_\Omega \hat{W}(\nabla u(x))\mathrm{d}x, \tag{1.6}$$

with

$$\hat{W}(\lambda) = \inf_{\psi \in W^{1,p}_{per}(Y)} \int_Y W(y, \lambda + \nabla\psi(y))\mathrm{d}y. \tag{1.7}$$

Here Y denotes the unit cube and $W^{1,p}_{per}(Y)$ is the $W^{1,p}$-closure of all the C^1 functions which are periodic on the unit cube.

Definition 2

Ω is **strongly star-shaped** if $\exists\, x_0 \in \Omega$ such that

$$\overline{-x_0 + \Omega} \subset \alpha(-x_0 + \Omega) \quad \forall \alpha > 1. \tag{1.8}$$

Theorem 1.2

Assume that Ω is a bounded $C^{0,1}$ domain in $\mathbb{R}^{\mathrm{N}}$ and strongly star-shaped. Assume furthermore that W satisifes
(i) $W(y,\lambda)$ is convex with respect to λ,
(ii)$a|\lambda|^p \leq W(y,\lambda) \leq M(\lambda) \quad \forall y$, where $\mathrm{N} < p < \infty$, $a > 0$ and $M(\lambda)$ is finite for all finite λ. Then $\{I^\varepsilon\}_{\varepsilon>0}$ given by (1.1) is Γ-convergent to I given by (1.6) and (1.7).

Proof of theorem 1.2

The proof falls into two parts: the demonstration of a lower bound and the explicit construction of a sequence which achieves the bound. We state the conclusions separately:

Lemma 1.3

Let the assumptions of Theorem 1.2 *hold, let* $\{I^\varepsilon\}_{\varepsilon>0}$ *and* I *be given by* (1.1) *and* (1.6), *respectively. Let* $u^\varepsilon \rightharpoonup u$ *in* $W^{1,p}(\Omega)$*. Then*

$$\liminf_{\varepsilon\to 0} I^\varepsilon(u^\varepsilon) \geq I(u).$$

Lemma 1.4

Let the assumptions of Theorem 1.2 *hold, let* $\{I^\varepsilon\}_{\varepsilon>0}$ *and* I *be given by* (1.1) *and* (1.6), *respectively. Then there is a sequence* $\{u^\varepsilon\}_{\varepsilon>0}$ *with* $u^\varepsilon \rightharpoonup u$ *in* $W^{1,p}(\Omega)$ *and*

$$\lim_{\varepsilon\to 0} I^\varepsilon(u^\varepsilon) = I(u).$$

To establish the lower bound (Lemma 1.3) we approximate the integrand W by a nondecreasing sequence W^m of convex integrands which satisfy polynomial growth conditions. The following special case of a result of MARCELLINI applies to such integrands.

Theorem 1.5

(MARCELLINI) Let Ω be a bounded $C^{0,1}$ domain and assume that
(i) $W(y,\lambda)$ is convex in λ,
(ii) $a|\lambda|^p \le W(y,\lambda) \le b(1+|\lambda|^p), \quad a>0, \quad p>1]$. Then $\{I^\varepsilon\}_{\varepsilon>0}$ given by (1.1) is Γ-convergent to I given by (1.6).

We will use the following standard approximation lemma.

Lemma 1.6

Let W be as in Theorem 1.2. Then there is a non-decreasing sequence of functions $W^m(y,\lambda)$ such that (i) $W(y,\lambda)$ is convex with respect to λ,
(ii)$W(y,\lambda) = \lim_{m\to\infty} W^m(y,\lambda)$,
(iii) $a|\lambda|^p \le W(y,\lambda) \le b(1+|\lambda|^p), \quad a>0, \quad p>1]$.

Proof.

Define

$$\tilde{W}^m(y,\lambda) := \sup_{|\mu|\leq m} \left\{ W(y,\mu) + \frac{\partial}{\partial\mu} W(y,\mu)\cdot(\lambda-\mu) \right\},$$

where $\frac{\partial}{\partial\mu} W(y,\mu)$ denotes any subdifferential at μ. Clearly, $\tilde{W}^m$ is nondecreasing in m and each $\tilde{W}^m(y,\lambda)$ is convex in λ as a supremum over affine functions. By convexity of W we have $\tilde{W}^m(y,\lambda) = W(y,\lambda)$ if $|\lambda| \leq m$ which proves (ii). Each $\tilde{W}^m$ grows linearly at infinity. The rate of growth is bounded by $\left[\frac{\partial}{\partial\mu} W(y,\mu)\right]$ which is uniformly bounded for $|\mu| \leq m$ by the following argument. Let $B(\lambda) = \sup_y W(y,\lambda)$. The function B is convex as a supremum over convex functions and finite by assumption, therefore continuous on $\mathbb{R}^{m\mathrm{N}}$. Now we have for every unit vector $e \in \mathbb{R}^{m\mathrm{N}}$

Proof (Cont.)

$$W(y,\mu) - W(y,\mu - e) \leq \frac{\partial}{\partial \mu} W(y,\mu) \cdot e \leq W(y,\mu + e) - W(y,\mu).$$

It follows that

$$\left| \frac{\partial}{\partial \mu} W(y,\mu) \cdot e \right| \leq B(\mu - e) + B(\mu) + B(\mu + e) \leq C_m$$

$\forall |\mu| \leq m$, since $|\mu \pm e| \leq m + 1$ and B is continuous. Thus we have $\left| \frac{\partial}{\partial \mu} W(y,\mu)| \right| \leq C_m$ if $|\mu| \leq m$. Now $W(m,\lambda) = \max\{\tilde{W}^m(y,\lambda), a|\lambda|^p\}$, satisies (i), (ii) and (iii). □

Lemma 1.7

Let W and W^m be as in Theorem 1.2 and lemma 1.6. Let for a fixed $\lambda \in \mathrm{M}^{m\times \mathrm{N}}$

$$J^m(\psi) = \int_Y W^m(y, \lambda + \nabla\psi(y))\mathrm{d}y.$$

Then $\tilde{W}^m(\lambda) = \inf_{\psi \in W^{1,p}_{per}(Y)} J^m(\psi)$ *converges to*

$$\tilde{W}(\lambda) = \inf_{\psi \in W^{1,p}_{per}(Y)} J(\psi) \text{ as } m \to \infty.$$

Proof.

Note that the infimum of J^m is attained by the convexity and coercivity assumptions and let ψ^m be a minimizer of J^m. Clearly $J^m(\psi^m) \le J^m(0) \le W(\lambda)$, so by coercivity we may assume that ψ^m is equibounded in $W^{1,p}(Y)$ and we can extract a weakly convergent subsequence which we again denote by ψ^m. Let ψ be the limit. We wish to show that ψ minimizes J. Let $\phi \in W^{1,p}_{per}(Y)$ be arbitrary.

Proof (Cont.)

Becasue J^m is non-decreasing, we obtain for every fixed $l \in \mathbb{N}$

$$J(\phi) = \lim_{m\to\infty} J^m(\phi) \geq \liminf_{m\to\infty} J^m(\phi^m) \geq \liminf_{m\to\infty} J^l(\phi^m) \geq J^l(\phi),$$

where we used the weak lower semicontinuity of J^l (note that W^l is convex) in the last step. Letting $l \to \infty$ and using the monotone convergence theorem we see thatϕ minimizes J and

$$\tilde{W}^m(\lambda) = J^m(\phi^m) \to J(\phi) = \tilde{W}(\lambda). \quad \square$$

Proof of lemma 1.3.

Let $u^\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega)$. Then

$$\liminf_{\varepsilon\to 0} I^\varepsilon(u^\varepsilon) \geq \liminf_{\varepsilon\to 0} \int_\Omega W^m\left(\frac{x}{\varepsilon}, \nabla u^\varepsilon(x)\right) \mathrm{d}x \geq \int_\Omega \tilde{W}^m(\nabla u)\mathrm{d}x$$

by Theorem 1.5. The conclusion follows from Lemma 1.7 by letting $m \to \infty$ and using the monotone convergence theorem. $\square$

Lemma 1.8

Let Ω be a bounded $C^{0,1}$ domain which is strongly star-shaped. Let $W : \mathbb{R}^{m\mathrm{N}} \to \mathbb{R}_0^+$ be convex and $u \in W^{1,p}(\Omega)$ such that

$$\int_\Omega W(\nabla u)\mathrm{d}x < \infty.$$

Then
(a) there is a sequence $v_n \in C^\infty(\overline{\Omega})$ such that $v_n \to u$ in $W^{1,p}(\Omega)$ and

$$\int_\Omega W(\nabla v_n)\mathrm{d}x \to \int_\Omega W(\nabla u)\mathrm{d}x.$$

(b)We can find a sequence of piecewise affine functions with the same properties.

Lemma 1.9

Assume $\varepsilon, \delta > 0$, $f : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R} \cup (\pm\infty)$. Then there is a mapping $\varepsilon \to \delta(\varepsilon)$ such that $\varepsilon \to 0$ implies $\delta(\varepsilon) \to 0$ and

$$\limsup_{\varepsilon\to 0} f(\delta(\varepsilon), \varepsilon) \le \limsup_{\delta\to 0} \limsup_{\varepsilon\to 0} f(\delta, \varepsilon)$$

We turn to the proof of Lemma 1.4. We proceed in three steps, considering u to be affine or piecewise affine in steps one and two.

Proof of lemma 1.4.

Step 1. u affine, says $u = \lambda x$. Let $\psi \in W^{1,p}_{per}(Y)$ be the minimizer of $\int_Y W(y, \lambda + \nabla\psi(y))\mathrm{d}y$ and set $u^\varepsilon(x) = \lambda x + \varepsilon\psi\left(\frac{x}{\varepsilon}\right)$. By periodicity

$$\int_\Omega W\left(\frac{x}{\varepsilon}, \nabla u^\varepsilon(x)\right)\mathrm{d}x \to |\Omega|\cdot\int_Y W(y, \lambda+\nabla\psi(y))\mathrm{d}y = ||\Omega|\tilde{W}(\lambda),$$

which is the desired result.

Step 2. Next we consider the case that u is piecewise affine, i.e. there are finitely many disjoint open sets Ω_i and a set N of measure zero such that $\Omega = \bigcup_{i=1}^{N} \Omega_i \cup N$ and $\nabla u \,|_{\Omega_i} = \lambda_i$. Let $\psi_i \in W^{1,p}_{per}(Y)$ be the minimizer of $\int_Y W(y, \lambda_i + \nabla\psi(y))\mathrm{d}y$ and define $v^\varepsilon(x) = \lambda x + \varepsilon\psi_i\left(\frac{x}{\varepsilon}\right)$ for $x \in \Omega_i$. We define

$$\Sigma^\delta = \{x \in \Omega \mid dist\left(x, \bigcup_i \partial\Omega\right) \leq \delta\}.$$

Proof (Cont.)

Choose $\phi^\delta \in W^{1,\infty}(\Omega)$ such that $0 \le \phi^\delta \le 1$ and
$\phi^\delta(x) = \begin{cases} 1 & \text{if } x \in \Sigma^\delta \\ 0 & \text{if } x \in \Omega \setminus \Sigma^{2\delta} \end{cases}$. We set $v^{\varepsilon,\delta} = (1-\phi^\delta)v^\varepsilon + \phi^\delta u$.
Now, $v^{\varepsilon,\delta} = u$ in Σ^δ and hence $v^{\varepsilon,\delta} \in W^{1,p}(\Omega)$. In Ω_i we have

$$\nabla v^{\varepsilon,\delta} = (1-\phi^\delta)\nabla v^\varepsilon + \phi^\delta \lambda_i + \nabla \phi^\delta (u - v^\varepsilon);$$

so for $0 < t < 1$,
$I^\varepsilon(tv^{\varepsilon,\delta};\Omega_i) =$
$\int_{\Omega_i} W\left(\frac{x}{\varepsilon}, t(1-\phi^\delta)\nabla v^\varepsilon + t\phi^\delta\lambda_i + (1-t)\frac{t}{1-t}\nabla\phi^\delta(u-v^\varepsilon)\right)\mathrm{d}x.$
Now, $t(1-\phi^\delta) + t\phi^\delta + 1 - t = 1$; hence by convexity
$I^\varepsilon(tv^{\varepsilon,\delta};\Omega_i) \le \int_{\Omega_i} t(1-\phi^\delta)W\left(\frac{x}{\varepsilon}, \nabla v^\varepsilon\right)\mathrm{d}x +$
$\int_{\Omega_i} t\phi^\delta W\left(\frac{x}{\varepsilon}, \lambda_i\right)\mathrm{d}x + (1-t)\int_{\Omega_i} t\phi^\delta W\left(\frac{x}{\varepsilon}, \frac{t}{1-t}\nabla\phi^\delta \varepsilon\psi\left(\frac{x}{\varepsilon}\right)\right)\mathrm{d}x$
$= I_1 + I_2 + I_3$. We pass the limit in the order of $\epsilon \to 0$, $\delta \to 0$, $t \to 1$. As above, $W\left(\frac{x}{\varepsilon}, \nabla v^\varepsilon\right) \to \tilde{W}(\lambda_i)$ in $L^1(\Omega_i)$, therefore

$$\lim_{t\to 1}\lim_{\delta\to 0}\lim_{\varepsilon\to 0} I_2 = 0$$

Proof (Cont.)

$$\lim_{t\to 1}\lim_{\delta\to 0}\limsup_{\varepsilon\to 0} I^{\varepsilon}(tv^{\varepsilon,\delta};\Omega_i) \leq |\Omega_i|\tilde{W}(\lambda_i);$$

by summing over i,

$$\lim_{t\to 1}\lim_{\delta\to 0}\limsup_{\varepsilon\to 0} I^{\varepsilon}(tv^{\varepsilon,\delta};\Omega_i) \leq I(u).$$

Using lemma 1.9, we obtain $\delta(\varepsilon), t(\varepsilon)$ and $\delta(\varepsilon)\to 0, t(\varepsilon)\to 1$ as $\varepsilon\to 0$, such that with $u^{\varepsilon} = t(\varepsilon)v^{\varepsilon,\delta(\varepsilon)}$,

$$\lim_{t\to 1}\lim_{\delta\to 0}\limsup_{\varepsilon\to 0} I^{\varepsilon}(u^{\varepsilon}) \leq I(u). \tag{1.9}$$

Furthermore $u^{\varepsilon}\to u$ in $L^p(\Omega)$ since $v^{\varepsilon,\delta}\to u$ in $L^p(\Omega)$ uniformly in δ. By equicoerciveness $\|\nabla u^{\varepsilon}\|_{L^p}$ is equibounded, so $u^{\varepsilon}\rightharpoonup u$ in $W^{1,p}(\Omega)$. By Lemma 1.3 we can replace $\limsup_{\varepsilon\to 0}$ by $\lim_{\varepsilon\to 0}$. Note that we used $|\partial\Omega_i| = 0$.

Proof (Cont.)

Step 3. Now we consider a general $u \in W^{1,p}(\Omega)$. It is sufficient to consider $I(u) = \int_\Omega \tilde{W}(\nabla u)\mathrm{d}x < \infty$. We invoke Lemma 1.8 to obtain a sequence u_i of of piecewise affine functions such that $u_i \to u$ in $W^{1,p}(\Omega)$ and $I(u_i) \to I(u)$. To apply Lemma 1.8 we only have to check the convexity of $\tilde{W}$, which follows from the following calculation:
$t\tilde{W}(\lambda) + (1-t)\tilde{W}(\mu) =$
$\int_Y t \cdot W(y, \lambda + \nabla\psi_1(y))\mathrm{d}y + \int_Y (1-t) \cdot W(y, \lambda + \nabla\psi_2(y))\mathrm{d}y$
$\geq \int_Y \cdot W(y, t\lambda + (1-t)\mu + t\nabla\psi_1(y) + (1-t)t\nabla\psi_1(y)\nabla\psi_2(y))\mathrm{d}y$
$\geq \tilde{W}(t\lambda + (1-t)\mu)$.
Using Step 2 we find, $u_i^\varepsilon \to u_i$ in $L^p(\Omega)$ such that $I^\varepsilon(u_i^\varepsilon) \to I(u_i)$. This gives

$$\lim_{i\to\infty} \lim_{\varepsilon\to 0} \{|I^\varepsilon(u_i^\varepsilon) - I(u)| + \|u_i^\varepsilon - u\|_{L^p}\} = 0.$$

Applying once again Lemma 1.9 and using the equicoerciveness we can find a sequence $\{v^\varepsilon\}_{\varepsilon>0}$ such that $v^\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega)$ and $I^\varepsilon(v^\varepsilon) \to I(u)$. □

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Applying once again Lemma 1.9 and using the equicoerciveness we can find a sequence $\{v^\varepsilon\}_{\varepsilon>0}$ such that $v^\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega)$ and $I^\varepsilon(v^\varepsilon) \to I(u)$. □

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