Haar Measure and Fourier Transform on Locally Compact Abelian Groups

Anish Ray

Contents

1 Topological Groups

We begin with the basic topological structure underlying Haar measure. The key point is that the group operations must be compatible with the topology.

1.1 Definition and Properties

Definition 1.1. *A* topological group *is a group G equipped with a topology such that the maps*

$$G \times G \to G, \qquad (x, y) \mapsto xy,$$

and

$$G \to G, \qquad x \mapsto x^{-1},$$

are continuous.

We denote the identity element by 1. For $A, B \subset G$ and $x \in G$, write

$$Ax = \{yx : y \in A\}, \qquad xA = \{xy : y \in A\}, \qquad A^{-1} = \{y^{-1} : y \in A\},$$

and

$$AB = \{xy : x \in A,\ y \in B\}.$$

When $A = B$, we write $A^2 = AA$, not to be confused with $\{x^2 : x \in A\}$ (which is in general a proper subset of AA). We say that A is symmetric if $A = A^{-1}$. We also observe that $A \cap B = \varnothing$ if and only if $1 \notin A^{-1}B$.

Proposition 1.2. *Let G be a topological group.*

(a) *The topology of G is invariant under translations and inversion. In particular, if U is open, then xU, Ux, and U^{-1} are open for every $x \in G$. Moreover, AU and UA are open for every $A \subset G$.*

(b) *For every neighborhood U of 1, there is a symmetric neighborhood V of 1 such that*

$$V^2 \subset U.$$

(c) *If H is a subgroup of G, then $\overline{H}$ is also a subgroup.*

(d) *Every open subgroup of G is closed.*

(e) *If A and B are compact subsets of G, then AB is compact.*

Proof. For (a), left and right translations are homeomorphisms:

$$y \mapsto xy, \qquad y \mapsto yx,$$

with inverses given by translation by x^{-1}. Inversion is also a homeomorphism, since it is continuous and is its own inverse. Hence open sets remain open under translations and inversion. Finally,

$$AU = \bigcup_{a \in A} aU, \qquad UA = \bigcup_{a \in A} Ua,$$

so these are open whenever U is open.

For (b), continuity of multiplication at $(1, 1)$ gives neighborhoods V_1, V_2 of 1 such that $V_1V_2 \subset U$. Replacing both by

$$V = V_1 \cap V_1^{-1} \cap V_2 \cap V_2^{-1},$$

we obtain a symmetric neighborhood of 1 with $V^2 \subset U$.

For (c), the map $(x, y) \mapsto xy^{-1}$ is continuous. Since multiplication and inversion preserve subgroup membership, the closure of a subgroup is again stable under the operation $(x, y) \mapsto xy^{-1}$. Hence $\overline{H}$ is a subgroup.

For (d), if H is open, then every coset xH is open by (a). Since $G \setminus H$ is the union of all cosets different from H, it is open. Thus H is closed.

For (e), AB is the image of the compact set $A \times B$ under the continuous multiplication map $G \times G \to G$. Therefore AB is compact. $\square$

Remark. *This proposition contains the elementary facts that make analysis on groups possible. Translations behave topologically like the identity map, small neighborhoods can be multiplied inside prescribed neighborhoods, subgroup closures remain algebraic, open subgroups are automatically closed, and compact sets remain compact under multiplication. These facts will be used repeatedly when constructing Haar measure and later when defining convolution.*

1.2 Quotient Groups and Local Compactness

Suppose now that H is a subgroup of a topological group G. Let G/H denote the space of left cosets of H, and let

$$q : G \longrightarrow G/H, \qquad q(x) = xH,$$

be the quotient map. We equip G/H with the quotient topology, namely: a set $U \subset G/H$ is open if and only if $q^{-1}(U)$ is open in G.

Equivalently, since

$$q^{-1}(q(V)) = VH$$

for every $V \subset G$, and VH is open whenever V is open by Proposition 1.2, a set $U \subset G/H$ is open if and only if it has the form

$$U = q(V)$$

for some open set $V \subset G$.

Proposition 1.3. *Let H be a subgroup of the topological group G.*

(a) *If H is closed, then G/H is Hausdorff.*

(b) *If G is locally compact, then G/H is locally compact.*

(c) *If H is normal, then G/H is a topological group.*

Proof. For (a), suppose $xH \neq yH$. Then $y^{-1}x \notin H$. Since H is closed, there exists a neighborhood U of 1 such that

$$Uy^{-1}x \cap H = \varnothing.$$

Choose a symmetric neighborhood V of 1 with $V^2 \subset U$. Then

$$xH \quad \text{and} \quad yH$$

have disjoint neighborhoods $q(Vx)$ and $q(Vy)$ in G/H. Hence G/H is Hausdorff.

For (b), if K is a compact neighborhood of x in G, then $q(K)$ is compact and is a neighborhood of $q(x)$ in G/H. Thus G/H is locally compact.

For (c), assume H is normal. The group operations on G/H are well-defined by

$$(xH)(yH) = xyH, \qquad (xH)^{-1} = x^{-1}H.$$

Continuity follows directly from the quotient topology and the continuity of multiplication and inversion on G. ☐

Remark. *The role of the quotient topology is to make the projection map $q : G \to G/H$ continuous and open. Part* (a) *explains why closed subgroups are the correct subgroups to quotient by if one wants a reasonable Hausdorff space. Part* (c) *says that normality is precisely the condition needed for the quotient to inherit a group structure.*

Corollary 1.4. *Let G be a topological group.*

(a) *If G is T_1, then G is Hausdorff.*

(b) *If G is not T_1, then $\overline{\{1\}}$ is a closed normal subgroup of G, and $G/\overline{\{1\}}$ is a Hausdorff topological group.*

Proof. The set $\overline{\{1\}}$ is a closed subgroup by Proposition 1.2. It is in fact normal, since conjugation by any element of G is a homeomorphism sending 1 to 1. Therefore the quotient $G/\overline{\{1\}}$ is a Hausdorff topological group by Proposition 1.3.

If G is T_1, then $\{1\}$ is closed, so $\overline{\{1\}} = \{1\}$. Applying Proposition 1.3 to the trivial subgroup gives that G is Hausdorff. $\square$

From now on we shall assume, unless explicitly stated otherwise, that our topological groups are Hausdorff. In particular, every locally compact group will be assumed Hausdorff.

Proposition 1.5 (The open σ-compact subgroup)**.** *Every locally compact group G has a subgroup G_0 that is open, closed, and σ-compact.*

Proof. Let U be a symmetric compact neighborhood of 1. Define inductively

$$U^1 = U, \qquad U^{n+1} = U^n U.$$

Each U^n is compact by Proposition 1.2, because products of compact sets are compact. Now set

$$G_0 = \bigcup_{n=1}^{\infty} U^n.$$

Then G_0 is σ-compact.

We claim that G_0 is a subgroup. Since U is symmetric, each inverse of an element of U^n again lies in U^n. Also,

$$U^m U^n \subset U^{m+n}.$$

Thus G_0 is closed under multiplication and inversion.

Also, G_0 is open because it contains the neighborhood U of 1, and for any $x \in G_0$, the translate xU is an open neighborhood of x contained in G_0. Finally, every open subgroup is closed by Proposition 1.2. Hence G_0 is open, closed, and σ-compact. $\square$

Remark. *This proposition is a useful reduction device. Even if G itself is not σ-compact, every locally compact group contains a large open-and-closed σ-compact subgroup. In particular, the connected component of the identity is contained in such a subgroup, and when G is connected, G itself is σ-compact.*

1.3 Translations of Functions and Uniform Continuity

With the notation of Proposition 1.5, the group G is the disjoint union of the cosets of G_0. Since each coset is closed and open, G is homeomorphic to the product of G_0 with the discrete space G/G_0. In particular, if G is connected, then $G = G_0$, and hence G is σ-compact.

For a function f on a topological group G and for $y \in G$, define the left and right translates of f through y by

$$L_y f(x) := f(y^{-1}x), \text{ and } R_y f(x) := f(xy),$$

respectively. The choice of y^{-1} in the definition of L_y ensures the maps $y \mapsto L_y$ and $y \mapsto R_y$ are group homomorphisms:

$$L_y L_z = L_{yz}, \qquad R_y R_z = R_{zy}.$$

We say that f is *left uniformly continuous* if

$$\|L_y f - f\|_\infty \longrightarrow 0 \qquad \text{as } y \to 1.$$

Similarly, f is *right uniformly continuous* if

$$\|R_y f - f\|_\infty \longrightarrow 0 \qquad \text{as } y \to 1.$$

Remark. *The terminology "left" and "right" refers to the side on which the group element acts on the argument of f. Thus $L_y f(x) = f(y^{-1}x)$ measures uniform stability under left translation of the input, while $R_y f(x) = f(xy)$ measures stability under right translation.*

Proposition 1.6. *If $f \in C_c(G)$, then f is both left and right uniformly continuous.*

Proof. We prove right uniform continuity; the proof for left uniform continuity is similar. Let

$$K = \operatorname{supp}(f).$$

Since f is continuous and compactly supported, it is uniformly continuous on compact neighborhoods of K.

Let $\varepsilon > 0$. By continuity, for each $x \in K$ there exists a neighborhood U_x of 1 such that

$$|f(xy) - f(x)| < \varepsilon \qquad (y \in U_x).$$

Choose a neighborhood U of 1 small enough so that this estimate holds uniformly for all $x \in K$ and all $y \in U$. More explicitly, one covers the compact set K by finitely many such neighborhoods and intersects the corresponding neighborhoods of 1.

If $x \in K$ and $y \in U$, then

$$|f(xy) - f(x)| < \varepsilon.$$

If $x \notin K$ and $xy \notin K$, then both terms vanish. Finally, by choosing U sufficiently small, the only remaining case is controlled by enlarging K slightly to a compact neighborhood of the support. Hence

$$\|R_y f - f\|_\infty < \varepsilon \qquad (y \in U).$$

Therefore

$$\|R_y f - f\|_\infty \to 0 \qquad \text{as } y \to 1.$$

Thus f is right uniformly continuous. The same argument, using $L_y f(x) = f(y^{-1}x)$, proves left uniform continuity. □

1.4 Examples of Locally Compact Groups

We briefly mention some standard examples of locally compact groups. These examples are useful because Haar measure is most naturally studied in this level of generality.

Example 1.7 (Euclidean spaces). *The group $\mathbb{R}^n$, with addition and its usual topology, is a locally compact group. More generally, every finite-dimensional Lie group is locally compact. In particular, matrix groups such as*

$$\mathrm{GL}(n, \mathbb{R})$$

are locally compact groups.

Example 1.8 (Classical compact groups). *The unit circle*

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$$

is a compact group under multiplication. More generally, compact Lie groups provide fundamental examples of locally compact groups.

Example 1.9 (Products and discrete groups). *Arbitrary products of compact groups are compact by Tychonoff's theorem, and hence locally compact. On the other hand, every discrete group is locally compact, since singletons are open and compact.*

Example 1.10 (Nonarchimedean examples). *Another important class of examples comes from local fields and matrix groups over them. In particular, the field of p-adic numbers $\mathbb{Q}_p$ is a locally compact field. The topology defined on this group is called the* p-adic *topology which is different from the usual topology on $\mathbb{R}$.*

2 Haar Measure

Let G be a locally compact group. We denote by $C_c(G)$ the space of compactly supported continuous functions on G, and by

$$C_c^+(G) = \{f \in C_c(G) : f \geq 0 \text{ and } f \neq 0\}.$$

Since the positive continuous functions on G separate points, the linear span of $C_c^+(G)$ is all of $C_c(G)$.

2.1 Definition and Properties

Definition 2.1 (Haar measure)**.** *A nonzero Radon measure μ on G is called a* left Haar measure *if*

$$\mu(xE) = \mu(E)$$

for every Borel set $E \subset G$ and every $x \in G$. Similarly, μ is called a right Haar measure *if*

$$\mu(Ex) = \mu(E)$$

for every Borel set $E \subset G$ and every $x \in G$.

Thus a left Haar measure is a measure that is invariant under left translation of sets, while a right Haar measure is invariant under right translation. In additive groups, such as $\mathbb{R}^n$, there is no distinction between left and right translations; in nonabelian groups the distinction becomes essential.

Proposition 2.2. *Let μ be a Radon measure on the locally compact group G, and define*

$$\widetilde{\mu}(E) = \mu(E^{-1})$$

for Borel sets $E \subset G$.

(a) *μ is a left Haar measure if and only if $\widetilde{\mu}$ is a right Haar measure.*

(b) *μ is a left Haar measure if and only if*

$$\int_G L_y f \, d\mu = \int_G f \, d\mu$$

for every $f \in C_c^+(G)$ and every $y \in G$.

Proof. For (a), observe that inversion exchanges left and right translation. Indeed,

$$(Ex)^{-1} = x^{-1}E^{-1}.$$

Therefore

$$\widetilde{\mu}(Ex) = \mu((Ex)^{-1}) = \mu(x^{-1}E^{-1}).$$

If μ is left invariant, then

$$\mu(x^{-1}E^{-1}) = \mu(E^{-1}) = \widetilde{\mu}(E),$$

so $\widetilde{\mu}$ is right invariant. The converse follows by applying the same argument to $\widetilde{\mu}$, since inversion is an involution.

For (b), recall that

$$L_y f(x) = f(y^{-1}x).$$

If μ is left invariant, then changing variables under the left translation $x \mapsto yx$ gives

$$\int_G L_y f \, d\mu = \int_G f \, d\mu.$$

Conversely, suppose this identity holds for every $f \in C_c^+(G)$ and every $y \in G$. Then μ and the left translate of μ have the same integrals against all compactly supported continuous test functions. By the uniqueness part of the Riesz representation theorem, the two Radon measures are equal. Hence

$$\mu(yE) = \mu(E)$$

for every Borel set $E \subset G$. Thus μ is left Haar. □

Remark. *Part* (a) *shows that left and right Haar measures are formally equivalent through inversion. Thus it is enough to construct one of them. Folland chooses to work with left Haar measure. Part* (b) *gives the analytic form of left invariance: instead of checking all Borel sets, it is enough to check equality of integrals against functions in $C_c^+(G)$.*

The next theorem is the existence theorem: every locally compact group admits a left Haar measure. The proof is one of the fundamental constructions in the theory, but for our purposes we will state it and retain only the main idea.

Theorem 2.3 (Existence of left Haar measure)**.** *Every locally compact group G possesses a left Haar measure.*

Motivation and construction. We first describe the idea behind the construction. If $E \subset G$ and V is a nonempty open set, define

$$(E : V) = \inf \left\{ \#(A) : E \subset \bigcup_{x \in A} xV \right\},$$

where $\#(A)$ denotes the cardinality of A if A is finite, and $\#(A) = \infty$ otherwise. Thus $(E : V)$ measures how many left translates of V are needed to cover E.

If E_0 is a fixed precompact open set, then

$$\frac{(E : V)}{(E_0 : V)}$$

may be viewed as a rough estimate for the size of E, normalized so that E_0 has size 1. As V shrinks to the identity, one hopes that these "quasi-measures" converge to a left-invariant measure.

It is more convenient to carry this out using compactly supported continuous functions. For $f, \phi \in C_c^+(G)$, define the *Haar covering number*

$$(f : \phi) = \inf \left\{ \sum_{j=1}^{n} c_j : f \le \sum_{j=1}^{n} c_j L_{x_j} \phi,\ c_j > 0,\ x_j \in G,\ n \in \mathbb{N} \right\}.$$

This number is finite because $\operatorname{supp}(f)$ is compact and can be covered by finitely many left translates of the open set where ϕ is positive. Also,

$$(f : \phi) > 0, \qquad (f : \phi) \ge \frac{\|f\|_\infty}{\|\phi\|_\infty}.$$

The following elementary properties are the key input.

Lemma 2.4 (Properties of the Haar covering number)**.** *Let $f, g, \phi \in C_c^+(G)$. Then:*

(a)
$$(f : \phi) = (L_x f : \phi) \qquad \textit{for every } x \in G.$$

(b)
$$(cf : \phi) = c(f : \phi) \qquad \textit{for every } c > 0.$$

(c)
$$(f + g : \phi) \le (f : \phi) + (g : \phi).$$

(d)
$$(f : \phi) \le (f : g)(g : \phi).$$

Proof. The first identity follows from the equivalence

$$f \le \sum_j c_j L_{x_j} \phi \quad \Longleftrightarrow \quad L_x f \le \sum_j c_j L_{x x_j} \phi.$$

The homogeneity in (b) is immediate from the definition.

For (c), if

$$f \le \sum_{j=1}^{m} c_j L_{x_j} \phi \quad \text{and} \quad g \le \sum_{j=m+1}^{n} c_j L_{x_j} \phi,$$

then

$$f + g \leq \sum_{j=1}^{n} c_j L_{x_j} \phi.$$

Taking infima gives the desired inequality.

Finally, if

$$f \leq \sum_j c_j L_{x_j} g \quad \text{and} \quad g \leq \sum_k d_k L_{y_k} \phi,$$

then

$$f \leq \sum_{j,k} c_j d_k L_{x_j y_k} \phi.$$

Since

$$\sum_{j,k} c_j d_k = \left(\sum_j c_j \right) \left(\sum_k d_k \right),$$

taking infima gives

$$(f : \phi) \leq (f : g)(g : \phi).$$

□

Fix once and for all a function $f_0 \in C_c^+(G)$ and define

$$I_\phi(f) = \frac{(f : \phi)}{(f_0 : \phi)}, \qquad f, \phi \in C_c^+(G).$$

For each fixed ϕ, the functional I_ϕ is left-invariant, homogeneous, positive, and subadditive. Indeed, left-invariance follows from Lemma 2.4(a):

$$I_\phi(L_x f) = \frac{(L_x f : \phi)}{(f_0 : \phi)} = \frac{(f : \phi)}{(f_0 : \phi)} = I_\phi(f).$$

Thus I_ϕ resembles a positive linear functional, but it is not yet additive.

The quantities $I_\phi(f)$ should be viewed as normalized approximate integrals. They are not yet genuine integrals against a measure; rather, the proof constructs a limiting positive linear functional I, and the Riesz representation theorem then realizes I as integration against a Radon measure.

Moreover, by the preceding lemma,

$$(f_0 : f)^{-1} \leq I_\phi(f) \leq (f : f_0).$$

Indeed, applying part (d) gives

$$(f_0 : \phi) \leq (f_0 : f)(f : \phi),$$

which implies the lower bound, and

$$(f : \phi) \leq (f : f_0)(f_0 : \phi),$$

which implies the upper bound.

Thus, for each fixed $f \in C_c^+(G)$, the values $I_\phi(f)$ stay in a bounded interval independent of ϕ. The next step is to show that I_ϕ becomes approximately additive when the support of ϕ is sufficiently small.

Lemma 2.5 (Approximate additivity)**.** *Let $f_1, f_2 \in C_c^+(G)$ and let $\varepsilon > 0$. Then there exists a neighborhood V of 1 such that*

$$I_\phi(f_1) + I_\phi(f_2) \leq I_\phi(f_1 + f_2) + \varepsilon$$

whenever $\phi \in C_c^+(G)$ and $\operatorname{supp}(\phi) \subset V$.

Proof. Fix $g \in C_c^+(G)$ such that

$$g = 1 \qquad \text{on } \operatorname{supp}(f_1 + f_2),$$

and let $\delta > 0$ be chosen later. Define

$$h = f_1 + f_2 + \delta g, \qquad h_i = \frac{f_i}{h} \quad (i = 1, 2),$$

where we set $h_i = 0$ outside $\operatorname{supp}(f_i)$. Then

$$h_i \in C_c^+(G), \qquad h_1 + h_2 \le 1.$$

By Proposition 1.6, each h_i is right uniformly continuous. Hence there is a neighborhood V of 1 such that

$$|h_i(x) - h_i(y)| < \delta$$

for $i = 1, 2$ whenever

$$y^{-1}x \in V.$$

Now let $\phi \in C_c^+(G)$ with

$$\operatorname{supp}(\phi) \subset V.$$

Suppose

$$h \le \sum_{j=1}^{n} c_j L_{x_j} \phi.$$

Since

$$L_{x_j}\phi(x) = \phi(x_j^{-1}x),$$

the condition $x_j^{-1}x \in \operatorname{supp}(\phi)$ implies

$$x_j^{-1}x \in V.$$

Therefore,

$$|h_i(x) - h_i(x_j)| < \delta.$$

Using $f_i = hh_i$, we obtain

$$\begin{aligned} f_i(x) &= h(x)h_i(x) \\ &\le \sum_{j=1}^{n} c_j \phi(x_j^{-1}x) h_i(x) \\ &\le \sum_{j=1}^{n} c_j \phi(x_j^{-1}x)\big(h_i(x_j) + \delta\big). \end{aligned}$$

Thus

$$f_i \le \sum_{j=1}^{n} c_j \big(h_i(x_j) + \delta\big) L_{x_j}\phi.$$

By the definition of the covering number,

$$(f_i : \phi) \le \sum_{j=1}^{n} c_j \big(h_i(x_j) + \delta\big).$$

Adding the estimates for $i = 1, 2$, and using $h_1 + h_2 \le 1$, gives

$$\begin{aligned} (f_1 : \phi) + (f_2 : \phi) &\le \sum_{j=1}^{n} c_j \big(h_1(x_j) + h_2(x_j) + 2\delta\big) \\ &\le (1 + 2\delta) \sum_{j=1}^{n} c_j. \end{aligned}$$

Since the sum $\sum_j c_j$ can be chosen arbitrarily close to $(h : \phi)$, we get

$$(f_1 : \phi) + (f_2 : \phi) \le (1 + 2\delta)(h : \phi).$$

Dividing by $(f_0 : \phi)$ yields

$$I_\phi(f_1) + I_\phi(f_2) \le (1 + 2\delta) I_\phi(h).$$

But

$$h = f_1 + f_2 + \delta g,$$

so by subadditivity and homogeneity of I_ϕ,

$$I_\phi(h) \le I_\phi(f_1 + f_2) + \delta I_\phi(g).$$

Hence

$$I_\phi(f_1) + I_\phi(f_2) \le (1 + 2\delta)\big[I_\phi(f_1 + f_2) + \delta I_\phi(g)\big].$$

By the normalization estimate

$$I_\phi(f) \le (f : f_0),$$

we have

$$I_\phi(f_1 + f_2) \le (f_1 + f_2 : f_0), \qquad I_\phi(g) \le (g : f_0).$$

Therefore the excess over $I_\phi(f_1 + f_2)$ is bounded by

$$2\delta(f_1 + f_2 : f_0) + \delta(1 + 2\delta)(g : f_0).$$

Choose $\delta > 0$ sufficiently small so that

$$2\delta(f_1 + f_2 : f_0) + \delta(1 + 2\delta)(g : f_0) < \varepsilon.$$

Then

$$I_\phi(f_1) + I_\phi(f_2) \le I_\phi(f_1 + f_2) + \varepsilon,$$

whenever $\mathrm{supp}(\phi) \subset V$. This proves the lemma. □

Completion of the proof of Theorem 2.3. For each $f \in C_c^+(G)$, define the compact interval

$$X_f = \left[(f_0 : f)^{-1}, (f : f_0)\right].$$

Let

$$X = \prod_{f \in C_c^+(G)} X_f.$$

By Tychonoff's theorem, X is compact. By the estimate

$$(f_0 : f)^{-1} \le I_\phi(f) \le (f : f_0),$$

each functional I_ϕ determines a point of X.

For each compact neighborhood V of 1, let

$$K(V) = \overline{\{I_\phi : \mathrm{supp}(\phi) \subset V\}}^{X}.$$

If $V_1, \ldots, V_n$ are compact neighborhoods of 1, then

$$\bigcap_{j=1}^n K(V_j) \supset K\left(\bigcap_{j=1}^n V_j\right).$$

Thus the family $\{K(V)\}$ has the finite intersection property. Since X is compact, there exists

$$I \in \bigcap_V K(V),$$

where V ranges over compact neighborhoods of 1.

This means that for every neighborhood V of 1, every finite collection $f_1, \dots, f_n \in C_c^+(G)$, and every $\varepsilon > 0$, there exists $\phi \in C_c^+(G)$ with

$$\mathrm{supp}(\phi) \subset V$$

such that

$$|I(f_j) - I_\phi(f_j)| < \varepsilon, \qquad j = 1, \dots, n.$$

We now pass the algebraic properties of I_ϕ to the limit. From the properties of the covering number, each I_ϕ is left-invariant, positive, homogeneous, and subadditive. Hence I is also left-invariant, positive, homogeneous, and subadditive on $C_c^+(G)$.

The approximate additivity lemma gives the reverse inequality needed for additivity. Namely, for $f, g \in C_c^+(G)$ and $\varepsilon > 0$, choose V so small that

$$I_\phi(f) + I_\phi(g) \leq I_\phi(f + g) + \varepsilon$$

whenever $\mathrm{supp}(\phi) \subset V$. Approximating I by such I_ϕ on the finite set $\{f, g, f + g\}$ and then letting the error tend to zero gives

$$I(f) + I(g) \leq I(f + g).$$

Since subadditivity already gives

$$I(f + g) \leq I(f) + I(g),$$

we have

$$I(f + g) = I(f) + I(g).$$

Together with homogeneity, this gives

$$I(af + bg) = aI(f) + bI(g)$$

for all $f, g \in C_c^+(G)$ and $a, b > 0$.

Now extend I to all of $C_c(G)$ by writing

$$f = f^+ - f^-,$$

where

$$f^+ = \max(f, 0), \qquad f^- = \max(-f, 0),$$

and setting

$$I(f) = I(f^+) - I(f^-).$$

The additivity on $C_c^+(G)$ ensures that this extension is well-defined and linear. It is also positive, since $f \geq 0$ implies $I(f) \geq 0$.

Therefore I is a positive left-invariant linear functional on $C_c(G)$. This positive linear functional I is the Haar integral. By the Riesz representation theorem, there exists a Radon measure μ on G such that

$$I(f) = \int_G f \, d\mu \qquad (f \in C_c(G)).$$

Since I is left-invariant,

$$\int_G L_y f \, d\mu = \int_G f \, d\mu$$

for every $f \in C_c^+(G)$ and every $y \in G$. By Proposition 2.2, μ is a left Haar measure. This proves the theorem. □

Proposition 2.6 (Positivity of Haar measure). *If λ is a left Haar measure on G, then*

$$\lambda(U) > 0$$

for every nonempty open set $U \subset G$, and

$$\int_G f \, d\lambda > 0$$

for every $f \in C_c^+(G)$.

Proof. Suppose, for contradiction, that $U \subset G$ is nonempty and open, but

$$\lambda(U) = 0.$$

Then, by left invariance,

$$\lambda(xU) = \lambda(U) = 0 \qquad \text{for every } x \in G.$$

Let $K \subset G$ be compact. Since U is nonempty, the translates of U cover G, hence also cover K:

$$K \subset \bigcup_{x \in G} xU.$$

By compactness, finitely many translates suffice:

$$K \subset \bigcup_{j=1}^{n} x_j U.$$

Therefore

$$\lambda(K) \leq \sum_{j=1}^{n} \lambda(x_j U) = 0.$$

Thus every compact set has λ-measure zero. Since λ is a Radon measure, it follows that λ is identically zero, contradicting the definition of Haar measure as a nonzero Radon measure. Hence

$$\lambda(U) > 0$$

for every nonempty open set U.

Now let $f \in C_c^+(G)$. Since f is nonzero and continuous, there exists $c > 0$ such that

$$U = \{x \in G : f(x) > c\}$$

is nonempty and open. Then

$$\int_G f \, d\lambda \geq \int_U f \, d\lambda \geq c \, \lambda(U) > 0.$$

Therefore

$$\int_G f \, d\lambda > 0.$$

□

Proposition 2.7. *Let μ be a left Haar measure on G. Then*

$$\mu(G) < \infty \qquad \Longleftrightarrow \qquad G \textit{ is compact.}$$

Proof. If G is compact, then

$$\mu(G) < \infty$$

because μ is a Radon measure.

Conversely, suppose G is not compact. Let V be a compact neighborhood of 1. Since G is not compact, it cannot be covered by finitely many left translates of V. Hence we can choose a sequence $\{x_n\}_{n=1}^{\infty}$ in G such that

$$x_n \notin \bigcup_{j=1}^{n-1} x_j V \qquad (n \geq 2).$$

By Proposition 1.2, there exists a symmetric neighborhood U of 1 such that

$$UU \subset V.$$

We claim that the sets

$$x_n U, \qquad n = 1, 2, \ldots,$$

are pairwise disjoint. Indeed, suppose $m > n$ and

$$x_n U \cap x_m U \neq \varnothing.$$

Then there exist $u, v \in U$ such that

$$x_n u = x_m v.$$

Therefore

$$x_m = x_n u v^{-1}.$$

Since U is symmetric, $v^{-1} \in U$, and hence

$$uv^{-1} \in UU \subset V.$$

Thus

$$x_m \in x_n V,$$

contradicting the choice of x_m.

Therefore the sets $x_n U$ are pairwise disjoint. By left invariance,

$$\mu(x_n U) = \mu(U) \qquad \text{for all } n.$$

By Proposition 2.6, since U is a nonempty open set,

$$\mu(U) > 0.$$

Hence

$$\mu(G) \geq \mu\left(\bigcup_{n=1}^{\infty} x_n U\right) = \sum_{n=1}^{\infty} \mu(x_n U) = \sum_{n=1}^{\infty} \mu(U) = \infty.$$

Thus, if G is not compact, then $\mu(G) = \infty$. This proves the equivalence. $\square$

Theorem 2.8 (Uniqueness of left Haar measure)**.** *If λ and μ are left Haar measures on G, then there exists a constant $c > 0$ such that*

$$\mu = c\lambda.$$

Proof. By Proposition 2.6, for every $f \in C_c^+(G)$,

$$\int_G f\, d\lambda > 0, \qquad \int_G f\, d\mu > 0.$$

It is enough to show that the ratio

$$\frac{\int_G f\, d\lambda}{\int_G f\, d\mu}$$

is independent of $f \in C_c^+(G)$.

Fix $f, g \in C_c^+(G)$. Choose a symmetric compact neighborhood V of 1. Set

$$A = (\operatorname{supp} f)V \cup V(\operatorname{supp} f), \qquad B = (\operatorname{supp} g)V \cup V(\operatorname{supp} g).$$

Then A and B are compact. Since λ and μ are Radon measures,

$$\lambda(A), \lambda(B), \mu(A), \mu(B) < \infty.$$

By Proposition 1.6, choosing V sufficiently small, we may assume that

$$|f(xy) - f(x)| < \varepsilon \qquad \text{and} \qquad |g(xy) - g(x)| < \varepsilon$$

whenever $y \in V$. Choose $h \in C_c^+(G)$ such that

$$h(x) = h(x^{-1}) \qquad \text{and} \qquad \operatorname{supp}(h) \subset V.$$

Using left invariance of λ and μ, together with Fubini's theorem, we compute

$$\int_G h\,d\mu \int_G f\,d\lambda = \iint_{G\times G} h(y)f(x)\,d\lambda(x)\,d\mu(y),$$

while

$$\int_G h\,d\lambda \int_G f\,d\mu = \iint_{G\times G} h(y)f(yx)\,d\lambda(x)\,d\mu(y).$$

Therefore

$$\begin{aligned}&\left|\int_G h\,d\mu \int_G f\,d\lambda - \int_G h\,d\lambda \int_G f\,d\mu\right| \\ &\qquad = \left|\iint h(y)\big(f(x)-f(yx)\big)\,d\lambda(x)\,d\mu(y)\right|.\end{aligned}$$

Since h is supported in V, the integrand is supported inside the compact set A in the x-variable, and by the choice of V we obtain

$$\left|\int h\,d\mu \int f\,d\lambda - \int h\,d\lambda \int f\,d\mu\right| \le \varepsilon\,\lambda(A)\int h\,d\mu.$$

Similarly,

$$\left|\int h\,d\lambda \int g\,d\mu - \int h\,d\mu \int g\,d\lambda\right| \le \varepsilon\,\mu(B)\int h\,d\lambda.$$

Dividing the first inequality by

$$\int h\,d\mu \int f\,d\mu$$

and the second by

$$\int h\,d\lambda \int g\,d\mu,$$

and then adding the two resulting inequalities, we get

$$\left|\frac{\int f\,d\lambda}{\int f\,d\mu} - \frac{\int g\,d\lambda}{\int g\,d\mu}\right| \le \varepsilon\left(\frac{\lambda(A)}{\int f\,d\mu} + \frac{\mu(B)}{\int g\,d\mu}\right).$$

Since $\varepsilon > 0$ is arbitrary, the two ratios are equal. Therefore there exists a constant $c > 0$ such that

$$\int_G f\,d\mu = c\int_G f\,d\lambda \qquad (f \in C_c^+(G)).$$

By linearity, the same equality holds for all $f \in C_c(G)$. Hence, by the uniqueness part of the Riesz representation theorem,

$$\mu = c\lambda.$$

□

2.2 Examples of Haar Measure

Some standard examples of Haar measure are:

Example 2.9 (Discrete groups)**.** *If G is a discrete group, then compact subsets of G are finite. A left and right Haar measure on G is the counting measure.*

Example 2.10 (Euclidean space)**.** *On the additive group $(\mathbb{R}^n, +)$, Haar measure is Lebesgue measure, up to multiplication by a positive constant. In particular, on $(\mathbb{R}, +)$, the Haar measure normalized by*

$$\mu([0,1]) = 1$$

is precisely Lebesgue measure.

Example 2.11 (The circle group). *Let*

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$$

with multiplication. The normalized Haar measure on $\mathbb{T}$ *is obtained from Lebesgue measure on* $[0, 2\pi]$ *under the parametrization*

$$t \longmapsto e^{it}.$$

Thus, for a Borel set $S \subset \mathbb{T}$,

$$\mu(S) = \frac{1}{2\pi}\lambda\big(\{t \in [0, 2\pi] : e^{it} \in S\}\big),$$

where λ *denotes Lebesgue measure. The factor* $(2\pi)^{-1}$ *normalizes the measure so that*

$$\mu(\mathbb{T}) = 1.$$

Example 2.12 (Positive real numbers under multiplication). *Consider the group*

$$\mathbb{R}_{>0}$$

under multiplication. A Haar measure is given by

$$d\mu(t) = \frac{dt}{t}.$$

Indeed, for an interval $[a, b] \subset \mathbb{R}_{>0}$,

$$\mu([a, b]) = \int_a^b \frac{dt}{t} = \log \frac{b}{a}.$$

Multiplying the interval by $g > 0$ *gives* $[ga, gb]$, *and*

$$\mu([ga, gb]) = \log \frac{gb}{ga} = \log \frac{b}{a}.$$

Thus the measure is invariant under multiplication.

Example 2.13 (Nonzero real numbers under multiplication). *For the multiplicative group*

$$\mathbb{R}^{\times} = \mathbb{R} \setminus \{0\},$$

a Haar measure is given by

$$d\mu(t) = \frac{dt}{|t|}.$$

This is the natural analogue of dt/t *on the two connected components* $\mathbb{R}_{>0}$ *and* $\mathbb{R}_{<0}$.

Example 2.14 (General linear group). *For*

$$G = \mathrm{GL}(n, \mathbb{R}),$$

viewed as an open subset of the vector space of all real $n \times n$ *matrices, a Haar measure is given by*

$$d\mu(X) = \frac{dX}{|\det X|^n},$$

where dX *denotes Lebesgue measure on* $\mathbb{R}^{n^2}$. *This follows from the change-of-variables formula and the fact that left multiplication by a fixed matrix rescales Lebesgue measure by a determinant factor.*

2.3 The Modular Function

We now discuss the relation between left and right Haar measures. Let μ be a left Haar measure on G. For $x \in G$, define a new measure μ_x by

$$\mu_x(E) = \mu(Ex)$$

for Borel sets $E \subset G$. Since left and right translations commute, μ_x is again a left Haar measure. Hence, by uniqueness of left Haar measure, there exists a number $\Delta(x) > 0$ such that

$$\mu_x = \Delta(x)\mu.$$

Equivalently,

$$\mu(Ex) = \Delta(x)\mu(E).$$

The function

$$\Delta : G \to (0, \infty)$$

is called the *modular function* of G.

Proposition 2.15 (The modular function)**.** *The modular function Δ is a continuous homomorphism from G into the multiplicative group of positive real numbers. Moreover, if μ is a left Haar measure on G, then for every $f \in L^1(\mu)$ and every $y \in G$,*

$$\int_G R_y f \, d\mu = \Delta(y^{-1}) \int_G f \, d\mu.$$

Equivalently,

$$\int_G f(xy) \, d\mu(x) = \Delta(y^{-1}) \int_G f(x) \, d\mu(x).$$

Proof. First, we prove that Δ is a homomorphism. For $x, y \in G$ and every Borel set $E \subset G$, we have

$$\Delta(xy)\mu(E) = \mu(Exy).$$

On the other hand,

$$\mu(Exy) = \Delta(y)\mu(Ex) = \Delta(y)\Delta(x)\mu(E).$$

Since μ is nonzero, it follows that

$$\Delta(xy) = \Delta(x)\Delta(y).$$

Thus Δ is a homomorphism from G to $(0, \infty)$.

Next, let E be a Borel set. Since

$$\chi_E(xy) = \chi_{Ey^{-1}}(x),$$

we get

$$\int_G \chi_E(xy) \, d\mu(x) = \mu(Ey^{-1}) = \Delta(y^{-1})\mu(E).$$

Therefore,

$$\int_G R_y \chi_E \, d\mu = \Delta(y^{-1}) \int_G \chi_E \, d\mu.$$

By linearity, the same identity holds for simple functions. By the usual approximation argument, it then holds for all $f \in L^1(\mu)$:

$$\int_G R_y f \, d\mu = \Delta(y^{-1}) \int_G f \, d\mu.$$

It remains to prove continuity of Δ. Choose $f \in C_c^+(G)$. Then

$$\int_G f \, d\mu > 0.$$

By the identity just proved,

$$\Delta(y^{-1}) = \frac{\int_G R_y f \, d\mu}{\int_G f \, d\mu}.$$

By Proposition 1.6, $R_y f \to f$ uniformly as $y \to 1$. Since all these functions have compact support contained in a fixed compact neighborhood for y sufficiently close to 1, it follows that

$$\int_G R_y f \, d\mu \longrightarrow \int_G f \, d\mu \qquad \text{as } y \to 1.$$

Hence

$$\Delta(y^{-1}) \to 1 \qquad \text{as } y \to 1.$$

Thus Δ is continuous at the identity. Since Δ is a homomorphism, continuity at the identity implies continuity everywhere. □

Definition 2.16 (Unimodular group)**.** *A locally compact group G is called* unimodular *if its modular function is identically* 1*, that is,*

$$\Delta(x) = 1 \qquad \textit{for every } x \in G.$$

Equivalently, every left Haar measure on G is also right invariant.

In particular, every abelian locally compact group is unimodular. More generally, many nonabelian groups are also unimodular. We record two useful criteria.

Let $[G, G]$ denote the smallest closed subgroup of G containing all commutators

$$[x, y] = xyx^{-1}y^{-1}.$$

This is called the *commutator subgroup* of G. It is a normal subgroup of G, and the quotient

$$G/[G, G]$$

is abelian. Moreover, $[G, G]$ is trivial precisely when G itself is abelian.

Proposition 2.17. *If $G/[G, G]$ is finite, then G is unimodular.*

Proof. The modular function

$$\Delta : G \to (0, \infty)$$

is a continuous homomorphism. Since $(0, \infty)$ is abelian, Δ kills commutators and therefore factors through the abelianization:

$$G \longrightarrow G/[G, G].$$

If $G/[G, G]$ is finite, then $\Delta(G)$ is a finite subgroup of $(0, \infty)$. But the multiplicative group $(0, \infty)$ has no nontrivial finite subgroups. Hence

$$\Delta(G) = \{1\},$$

so $\Delta \equiv 1$. Therefore G is unimodular. □

Proposition 2.18. *If G is compact, then G is unimodular.*

Proof. Let μ be a left Haar measure on G. For any $x \in G$,

$$Gx = G.$$

Therefore

$$\mu(G) = \mu(Gx) = \Delta(x)\mu(G).$$

Since G is compact, $\mu(G) < \infty$, and by positivity of Haar measure,

$$0 < \mu(G) < \infty.$$

Hence

$$\Delta(x) = 1$$

for every $x \in G$. Thus G is unimodular. □

We have already observed in Proposition 2.2 that if μ is a left Haar measure, then the measure

$$\widetilde{\mu}(E) = \mu(E^{-1})$$

is a right Haar measure. We now compute this right Haar measure in terms of μ and the modular function.

Proposition 2.19. *Let μ be a left Haar measure on G, and define*

$$\widetilde{\mu}(E) = \mu(E^{-1}).$$

Then

$$d\widetilde{\mu}(x) = \Delta(x)^{-1}\, d\mu(x).$$

Equivalently, for every $f \in C_c(G)$,

$$\int_G f(x)\, d\widetilde{\mu}(x) = \int_G f(x)\Delta(x)^{-1}\, d\mu(x).$$

Proof. Define a positive linear functional on $C_c(G)$ by

$$T(f) = \int_G f(x)\Delta(x)^{-1}\, d\mu(x).$$

We first show that T is right invariant. By Proposition 2.15,

$$\int_G R_y f\, d\mu = \Delta(y^{-1}) \int_G f\, d\mu.$$

Using the homomorphism property of Δ, we get

$$\Delta(xy)^{-1} = \Delta(y)^{-1}\Delta(x)^{-1}.$$

Hence

$$\begin{aligned} T(R_y f) &= \int_G f(xy)\Delta(x)^{-1}\, d\mu(x) \\ &= \Delta(y)^{-1} \int_G f(xy)\Delta(xy)^{-1}\, d\mu(x). \end{aligned}$$

Applying the change-of-variables formula from Proposition 2.15 gives

$$T(R_y f) = T(f).$$

Thus T is right invariant, and so the Radon measure associated to T is a right Haar measure.

By uniqueness of right Haar measure, there exists a constant $c > 0$ such that

$$\widetilde{\mu} = c\,\Delta^{-1}\mu.$$

It remains to show that $c = 1$.

Choose a symmetric neighborhood U of 1 on which

$$|\Delta(x)^{-1} - 1| < \frac{1}{2}|c^{-1} - 1|.$$

Then, since $U = U^{-1}$,

$$\widetilde{\mu}(U) = \mu(U).$$

On the other hand,

$$\widetilde{\mu}(U) = c \int_U \Delta(x)^{-1}\, d\mu(x).$$

Therefore,

$$|c^{-1} - 1|\mu(U) = \left| \int_U \left(\Delta(x)^{-1} - 1\right) d\mu(x) \right| < \frac{1}{2}|c^{-1} - 1|\mu(U).$$

Since $\mu(U) > 0$, this is impossible unless

$$c = 1.$$

Hence

$$d\widetilde{\mu}(x) = \Delta(x)^{-1}\, d\mu(x).$$

□

Corollary 2.20. *Left and right Haar measures are mutually absolutely continuous.*

Proof. The preceding proposition shows that a right Haar measure is obtained from a left Haar measure by multiplying by the positive continuous function

$$\Delta^{-1}.$$

Since

$$\Delta(x) > 0 \qquad \text{for all } x \in G,$$

the two measures have exactly the same null sets. Therefore they are mutually absolutely continuous. □

A Technical Convention

In full generality, locally compact groups need not be σ-compact, and Haar measure need not be σ-finite. This can lead to minor technical issues involving measurability, Fubini's theorem, and the precise duality between L^1 and L^∞.

For the purposes of this presentation, we shall avoid these complications by working only with the standard function spaces needed for Fourier analysis: $C_c(G)$ and $L^p(G)$ with respect to a fixed Haar measure. All functions appearing in convolution and Fourier transform will be assumed measurable, and all uses of Fubini–Tonelli will occur in settings where the relevant integrability hypotheses are satisfied.

Thus the technical refinements involving locally Borel sets, locally null sets, and locally measurable functions will not be needed here.

3 A Short Interlude on Banach Algebras

We recall a few notions from Banach algebra theory, since convolution will make $L^1(G)$ into a Banach algebra and the Fourier transform will later be interpreted as its Gelfand transform.

Definition 3.1 (Banach algebra)**.** *A* Banach algebra *is a complex Banach space A equipped with an associative multiplication such that*

$$\|xy\| \le \|x\|\,\|y\| \qquad (x, y \in A).$$

If there exists an element $1 \in A$ such that

$$1x = x1 = x \qquad (x \in A),$$

then A is called unital.

Definition 3.2 (Invertible elements and units)**.** *Let A be a unital Banach algebra. An element $x \in A$ is called* invertible *if there exists $y \in A$ such that*

$$xy = yx = 1.$$

The set of invertible elements of A is called the group of units *of A.*

Definition 3.3 (Ideals and maximal ideals)**.** *Let A be an algebra. A linear subspace $I \subset A$ is a* left ideal *if*

$$AI \subset I,$$

and a right ideal *if*

$$IA \subset I.$$

If both conditions hold, then I is called a two-sided ideal. *In a commutative algebra there is no distinction between left and right ideals.*

A proper ideal $M \subsetneq A$ is called maximal *if there is no proper ideal strictly between M and A.*

Definition 3.4 (Multiplicative linear functionals)**.** *Let A be a commutative Banach algebra. A nonzero linear functional*

$$h : A \to \mathbb{C}$$

is called multiplicative *if*

$$h(xy) = h(x)h(y) \qquad (x, y \in A).$$

The set of all multiplicative linear functionals on A is denoted by

$$\sigma(A),$$

and is called the spectrum *or* maximal ideal space *of A.*

The terminology is justified by the following standard fact: multiplicative linear functionals and maximal ideals are essentially the same objects.

Proposition 3.5 (Maximal ideals and characters)**.** *Let A be a commutative unital Banach algebra. Then the map*

$$h \longmapsto \ker h$$

gives a one-to-one correspondence between multiplicative linear functionals on A and maximal ideals of A.

Sketch. If h is multiplicative, then $\ker h$ is an ideal. Since $h(A) \subset \mathbb{C}$ is a nonzero subalgebra and $h(1) = 1$, the quotient

$$A/\ker h$$

is isomorphic to $\mathbb{C}$, so $\ker h$ is maximal.

Conversely, if M is a maximal ideal, then A/M is a complex Banach algebra with no nontrivial ideals. By the Gelfand–Mazur theorem, it is isomorphic to $\mathbb{C}$. The quotient map

$$A \to A/M \cong \mathbb{C}$$

therefore gives a multiplicative linear functional whose kernel is M. □

Definition 3.6 (Gelfand transform)**.** *Let A be a commutative Banach algebra and let*

$$\sigma(A)$$

be its maximal ideal space. For each $x \in A$, the Gelfand transform *of x is the function*

$$\widehat{x} : \sigma(A) \to \mathbb{C}$$

defined by

$$\widehat{x}(h) = h(x), \qquad h \in \sigma(A).$$

Thus the Gelfand transform sends an abstract algebra element to a function on the space of multiplicative linear functionals.

Remark. *This point of view is central for Fourier analysis on locally compact abelian groups. Once convolution is introduced, $L^1(G)$ becomes a commutative Banach algebra. Its multiplicative linear functionals will turn out to be exactly the continuous characters of G. Under this identification, the Gelfand transform of $f \in L^1(G)$ becomes the Fourier transform*

$$\widehat{f}(\gamma) = \int_G f(x)\overline{\gamma(x)}\, dx.$$

4 Convolution on Locally Compact Abelian Groups

From now on, let G be a locally compact abelian group, written additively, and let dx denote a fixed Haar measure on G. Since G is abelian, left and right Haar measures coincide. Also, we call a Borel measurable function on G to be a Borel function.

Definition 4.1. *For Borel functions f and g on G, their* convolution *is defined by*

$$(f * g)(x) = \int_G f(x-y)g(y)\,dy,$$

provided the integral exists. Equivalently, by translation-invariance of Haar measure,

$$(f * g)(x) = \int_G f(y)g(x-y)\,dy.$$

Thus convolution may be viewed as a weighted average of translates of f, where the weights are determined by g.

Theorem 4.2 (Basic properties of convolution)**.** *Let G be a locally compact abelian group.*

(a) *If the convolution exists, then*
$$f * g = g * f.$$

(b) *If $f \in L^1(G)$ and $g \in L^\infty(G)$, then $f * g$ exists for every $x \in G$, and*
$$\|f * g\|_\infty \le \|f\|_1 \|g\|_\infty.$$

(c) *If $f, g \in C_c(G)$, then $f * g \in C_c(G)$. More precisely, if*
$$\operatorname{supp}(f) \subset A, \qquad \operatorname{supp}(g) \subset B,$$
where A and B are compact, then
$$\operatorname{supp}(f * g) \subset A + B.$$

(d) *If $1 < p < \infty$, $1/p + 1/q = 1$, $f \in L^p(G)$, and $g \in L^q(G)$, then*
$$f * g \in C_0(G).$$

(e) *If $f, g \in L^1(G)$, then $f * g$ exists for almost every $x \in G$, belongs to $L^1(G)$, and satisfies*
$$\|f * g\|_1 \le \|f\|_1 \|g\|_1.$$

(f) *If $f, g, h \in L^1(G)$, then*
$$(f * g) * h = f * (g * h)$$
almost everywhere.

Proof. For (a), using the change of variables $y \mapsto x - y$, we get

$$(f * g)(x) = \int_G f(x-y)g(y)\,dy = \int_G f(y)g(x-y)\,dy = (g * f)(x).$$

For (b), if $f \in L^1(G)$ and $g \in L^\infty(G)$, then

$$\int_G |f(x-y)g(y)|\,dy \le \|g\|_\infty \int_G |f(x-y)|\,dy = \|g\|_\infty \|f\|_1.$$

Hence $f * g$ exists for every x and
$$\|f * g\|_\infty \le \|f\|_1 \|g\|_\infty.$$

For (c), assume $f, g \in C_c(G)$, with supports contained in compact sets A and B. If $x \notin A+B$, then there is no $y \in B$ such that $x-y \in A$. Hence

$$f(x-y)g(y) = 0$$

for all y, and therefore

$$(f * g)(x) = 0.$$

Thus

$$\operatorname{supp}(f * g) \subset A + B,$$

which is compact. Continuity follows from the uniform continuity of compactly supported continuous functions: if x' is close to x, then

$$|f(x'-y) - f(x-y)|$$

is uniformly small for all relevant y, and hence

$$(f * g)(x') \to (f * g)(x).$$

Therefore $f * g \in C_c(G)$.
For (d), Choose sequences $\{f_n\}$ and $\{g_n\}$ in $C_c(G)$ such that

$$\|f_n - f\|_p \to 0, \qquad \|g_n - g\|_q \to 0.$$

For each n, part (c) gives

$$f_n * g_n \in C_c(G) \subset C_0(G).$$

We claim that

$$f_n * g_n \to f * g$$

uniformly. Indeed, by Hölder's inequality, for every $x \in G$,

$$\begin{aligned} |(f * g)(x) - (f_n * g_n)(x)| &\le |((f - f_n) * g)(x)| + |f_n * (g - g_n)(x)| \\ &\le \|f - f_n\|_p \|g\|_q + \|f_n\|_p \|g - g_n\|_q. \end{aligned}$$

Since

$$\|f_n - f\|_p \to 0$$

and $\|f_n\|_p$ remains bounded, the right-hand side tends to 0, independently of x. Hence

$$\|f * g - f_n * g_n\|_\infty \to 0.$$

Since $C_0(G)$ is closed under uniform limits, and each $f_n * g_n \in C_0(G)$, we conclude that

$$f * g \in C_0(G).$$

For (e), by Fubini's theorem,

$$\int_G \int_G |f(x-y)g(y)|\, dy\, dx = \int_G |g(y)| \left(\int_G |f(x-y)|\, dx \right) dy.$$

By translation-invariance of Haar measure,

$$\int_G |f(x-y)|\, dx = \|f\|_1.$$

Thus

$$\int_G \int_G |f(x-y)g(y)|\, dy\, dx = \|f\|_1 \|g\|_1 < \infty.$$

Hence $f * g$ exists for almost every x, belongs to $L^1(G)$, and

$$\|f * g\|_1 \le \int_G \int_G |f(x-y)g(y)|\, dy\, dx = \|f\|_1 \|g\|_1.$$

Finally, (e) follows from Fubini's theorem. Indeed, for almost every x,

$$\begin{aligned}((f*g)*h)(x) &= \int_G (f*g)(x-z)h(z)\,dz \\ &= \int_G\int_G f(x-z-y)g(y)h(z)\,dy\,dz.\end{aligned}$$

By absolute integrability, Fubini's theorem allows us to interchange the order of integration. Rewriting the variables gives

$$((f*g)*h)(x) = \int_G f(x-u)(g*h)(u)\,du = (f*(g*h))(x).$$

Therefore convolution is associative on $L^1(G)$. $\square$

Remark. *Thus $L^1(G)$ is a Banach algebra under convolution:*

$$\|f*g\|_1 \le \|f\|_1\|g\|_1, \qquad (f*g)*h = f*(g*h).$$

Since G is abelian, this algebra is also commutative. If G is discrete, the Haar measure is counting measure and convolution becomes

$$(f*g)(x) = \sum_{y\in G} f(x-y)g(y).$$

In particular, for $G=\mathbb{Z}$ this is the usual convolution of sequences.

Theorem 4.3. *For any locally compact abelian group G, the space $L^1(G)$ is a commutative Banach algebra under convolution. If G is discrete, then $L^1(G)$ has a unit.*

Proof. By Theorem 4.2, if $f,g\in L^1(G)$, then $f*g$ exists almost everywhere, belongs to $L^1(G)$, and satisfies

$$\|f*g\|_1 \le \|f\|_1\|g\|_1.$$

Moreover, convolution is associative:

$$(f*g)*h = f*(g*h),$$

and since G is abelian, it is commutative:

$$f*g = g*f.$$

Thus $L^1(G)$ is a commutative Banach algebra under convolution.

Now suppose that G is discrete. Then Haar measure is counting measure, and convolution is given by

$$(f*g)(x) = \sum_{y\in G} f(x-y)g(y).$$

Let e be the function

$$e(0)=1, \qquad e(x)=0 \quad (x\neq 0).$$

Then $e\in L^1(G)$, and for every $f\in L^1(G)$,

$$(f*e)(x) = \sum_{y\in G} f(x-y)e(y) = f(x).$$

Similarly,

$$(e*f)(x) = f(x).$$

Hence e is the unit element of $L^1(G)$. $\square$

Remark. *If G is not discrete, then $L^1(G)$ has no unit. Indeed, the role of the identity would have to be played by a point mass at 0, but this is not an L^1 function with respect to Haar measure unless the group is discrete. Thus, for nondiscrete groups, one works instead with approximate identities.*

Theorem 4.4 (Translation in $(L^p(G))$). *Let G be a locally compact abelian group and let $1 \le p < \infty$. For $f \in L^p(G)$ and $x \in G$, define the translate*

$$f_x(y) = f(y - x).$$

Then

$$\|f_x\|_p = \|f\|_p \qquad (x \in G),$$

and the map

$$G \longrightarrow L^p(G), \qquad x \longmapsto f_x,$$

is uniformly continuous.

Proof. The equality

$$\|f_x\|_p = \|f\|_p$$

follows immediately from translation-invariance of Haar measure.

We now prove uniform continuity. Let $\varepsilon > 0$. Since $C_c(G)$ is dense in $L^p(G)$, choose $g \in C_c(G)$ such that

$$\|f - g\|_p < \frac{\varepsilon}{3}.$$

Let $K_0 = \mathrm{supp}(g)$. Choose a compact neighborhood W of 0. Then

$$K := K_0 \cup (K_0 + W)$$

is compact.

Since $g \in C_c(G)$, it is uniformly continuous. Hence there exists a neighborhood $V \subset W$ of 0 such that

$$\|g - g_x\|_\infty < \frac{\varepsilon}{3\, m(K)^{1/p}} \qquad (x \in V),$$

where m denotes Haar measure. For $x \in V$, the function $g - g_x$ is supported in K, and therefore

$$\|g - g_x\|_p \le m(K)^{1/p} \|g - g_x\|_\infty < \frac{\varepsilon}{3}.$$

Thus, for $x \in V$,

$$\begin{aligned} \|f - f_x\|_p &\le \|f - g\|_p + \|g - g_x\|_p + \|g_x - f_x\|_p \\ &= \|f - g\|_p + \|g - g_x\|_p + \|g - f\|_p \\ &< \varepsilon. \end{aligned}$$

Here we used translation-invariance of the L^p-norm in the equality

$$\|g_x - f_x\|_p = \|g - f\|_p.$$

Finally, for arbitrary $x, y \in G$,

$$f_x - f_y = (f - f_{y-x})_x.$$

Hence

$$\|f_x - f_y\|_p = \|f - f_{y-x}\|_p.$$

Therefore, if $y - x \in V$, then

$$\|f_x - f_y\|_p < \varepsilon.$$

This proves that $x \mapsto f_x$ is uniformly continuous from G into $L^p(G)$. $\square$

Theorem 4.5 (Approximation by convolution). *Given $f \in L^1(G)$ and $\varepsilon > 0$, there exists a neighborhood V of 0 in G with the following property: if u is a nonnegative Borel function which vanishes outside V, and if*

$$\int_G u(x)\, dx = 1,$$

then

$$\|f - f * u\|_1 < \varepsilon.$$

Proof. By Theorem 4.4, we can choose V so that

$$\|f - f_y\|_1 < \varepsilon \qquad \text{for all } y \in V,$$

where

$$f_y(x) = f(x - y).$$

If u satisfies the hypotheses, then

$$\begin{aligned}(f * u)(x) - f(x) &= \int_G f(x-y)u(y)\,dy - f(x)\int_G u(y)\,dy \\ &= \int_G \big[f(x-y) - f(x)\big]u(y)\,dy.\end{aligned}$$

Hence, by Fubini's theorem,

$$\begin{aligned}\|f * u - f\|_1 &\le \int_G |u(y)| \left(\int_G |f(x-y) - f(x)|\,dx\right) dy \\ &= \int_G \|f_y - f\|_1 u(y)\,dy.\end{aligned}$$

Since u vanishes outside V, $u \ge 0$, and $\int_G u(y)\,dy = 1$, we get

$$\|f * u - f\|_1 = \int_V \|f_y - f\|_1 u(y)\,dy < \varepsilon \int_V u(y)\,dy = \varepsilon.$$

Therefore

$$\|f - f * u\|_1 < \varepsilon.$$

□

5 The Dual Group

Let G be a locally compact abelian group, written additively.

Definition 5.1 (Character)**.** *A* character *of G is a continuous function*

$$\gamma : G \to \mathbb{T}$$

such that

$$\gamma(x + y) = \gamma(x)\gamma(y) \qquad (x, y \in G).$$

Here

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$$

is the unit circle.

The set of all continuous characters of G forms a group under pointwise multiplication. This group is denoted by

$$\Gamma = \widehat{G}$$

and is called the *dual group* of G. Thus, if

$$\gamma_1, \gamma_2 \in \Gamma,$$

then

$$(\gamma_1 + \gamma_2)(x) = \gamma_1(x)\gamma_2(x),$$

where the notation is additive on Γ but multiplicative in $\mathbb{T}$. The identity element of Γ is the constant character

$$0(x) = 1.$$

The inverse of a character is given by

$$(-\gamma)(x) = \gamma(x)^{-1} = \overline{\gamma(x)}.$$

It is often convenient to use the pairing notation

$$(x, \gamma) := \gamma(x).$$

With this notation, the defining properties become

$$(x + y, \gamma) = (x, \gamma)(y, \gamma),$$

and

$$(x, \gamma_1 + \gamma_2) = (x, \gamma_1)(x, \gamma_2).$$

Also,

$$(0, \gamma) = (x, 0) = 1,$$

and

$$(-x, \gamma) = (x, -\gamma) = \overline{(x, \gamma)}.$$

The next theorem explains the connection between characters of G and the Banach algebra structure of $L^1(G)$.

Theorem 5.2. *Let $\gamma \in \Gamma$. For $f \in L^1(G)$, define*

$$\widehat{f}(\gamma) = \int_G f(x)(-x, \gamma)\, dx.$$

Then, for each fixed $\gamma \in \Gamma$, the map

$$f \longmapsto \widehat{f}(\gamma)$$

is a nonzero complex homomorphism of the Banach algebra $L^1(G)$ into $\mathbb{C}$. Conversely, every nonzero complex homomorphism of $L^1(G)$ into $\mathbb{C}$ is of this form for a unique $\gamma \in \Gamma$.

Proof. Fix $\gamma \in \Gamma$. The map

$$f \longmapsto \widehat{f}(\gamma)$$

is clearly linear. We show that it is multiplicative with respect to convolution. If $f, g \in L^1(G)$, then by Fubini's theorem,

$$\begin{aligned}
\widehat{f * g}(\gamma) &= \int_G (f * g)(x)(-x, \gamma)\, dx \\
&= \int_G \int_G f(x - y)g(y)(-x, \gamma)\, dy\, dx.
\end{aligned}$$

Put $u = x - y$, so that $x = u + y$. Since γ is a character,

$$(-x, \gamma) = (-(u + y), \gamma) = (-u, \gamma)(-y, \gamma).$$

Therefore

$$\begin{aligned}
\widehat{f * g}(\gamma) &= \int_G \int_G f(u)g(y)(-u, \gamma)(-y, \gamma)\, du\, dy \\
&= \left(\int_G f(u)(-u, \gamma)\, du \right) \left(\int_G g(y)(-y, \gamma)\, dy \right) \\
&= \widehat{f}(\gamma)\widehat{g}(\gamma).
\end{aligned}$$

Hence $f \mapsto \widehat{f}(\gamma)$ is a complex homomorphism of $L^1(G)$ into $\mathbb{C}$.

Conversely, let

$$h : L^1(G) \longrightarrow \mathbb{C}$$

be a nonzero complex homomorphism. A standard result from Banach algebra theory says that every nonzero complex homomorphism on a Banach algebra is automatically a bounded linear functional; moreover, in this case

$$\|h\| = 1.$$

(In the nonunital case, the statement $|h\| = 1$ is understood through the standard extension of h to the unitization of $L^1(G)$.) Hence, by the duality

$$(L^1(G))^* = L^\infty(G),$$

there exists a function $\phi \in L^\infty(G)$, with

$$\|\phi\|_\infty = 1,$$

such that

$$h(f) = \int_G f(x)\phi(x)\,dx \qquad (f \in L^1(G)).$$

For $y \in G$, write

$$f_y(x) := f(x-y).$$

Let $f, g \in L^1(G)$. Using multiplicativity of h, we have

$$h(f)h(g) = h(f * g).$$

Using the representation of h by ϕ, the left-hand side is

$$h(f)h(g) = h(f)\int_G g(y)\phi(y)\,dy.$$

On the other hand,

$$\begin{aligned}
h(f * g) &= \int_G (f * g)(x)\phi(x)\,dx \\
&= \int_G \left(\int_G f(x-y)g(y)\,dy\right)\phi(x)\,dx \\
&= \int_G g(y)\left(\int_G f(x-y)\phi(x)\,dx\right)dy \\
&= \int_G g(y)h(f_y)\,dy.
\end{aligned}$$

Therefore

$$\int_G g(y)\big[h(f)\phi(y) - h(f_y)\big]\,dy = 0 \qquad (g \in L^1(G)).$$

Since this holds for every $g \in L^1(G)$, it follows that

$$h(f)\phi(y) = h(f_y)$$

for almost every $y \in G$.

Now choose $f \in L^1(G)$ such that

$$h(f) \neq 0.$$

For this fixed f, the map

$$y \longmapsto f_y$$

is continuous from G into $L^1(G)$, by the continuity of translations on $L^1(G)$. Since h is continuous, the map

$$y \longmapsto h(f_y)$$

is continuous. Hence

$$\phi(y) = \frac{h(f_y)}{h(f)}$$

almost everywhere. Replacing ϕ by this continuous representative, we may assume that ϕ is continuous. With this representative, the identity

$$h(f)\phi(y) = h(f_y)$$

holds for all $y \in G$. Indeed, both sides are continuous functions of y, and they agree almost everywhere; since every nonempty open set has positive Haar measure, they agree everywhere.

Thus, for every $f \in L^1(G)$ and every $y \in G$,

$$h(f)\phi(y) = h(f_y). \tag{1}$$

We now prove that ϕ is multiplicative. Replacing y by $x + y$ in (1), we get

$$h(f)\phi(x + y) = h(f_{x+y}). \tag{2}$$

On the other hand, applying (1) first to f_x and then to f, we get

$$h(f_{x+y}) = h((f_x)_y) = h(f_x)\phi(y) = h(f)\phi(x)\phi(y). \tag{3}$$

Comparing (2) and (3), and choosing f with $h(f) \neq 0$, gives

$$\phi(x + y) = \phi(x)\phi(y) \qquad (x, y \in G).$$

Since $\|\phi\|_\infty = 1$ and ϕ is continuous, we have

$$|\phi(x)| \leq 1 \qquad (x \in G).$$

The identity

$$\phi(x + y) = \phi(x)\phi(y)$$

also implies that ϕ is not identically zero, because otherwise $h = 0$. Hence

$$\phi(0) = 1.$$

Then

$$1 = \phi(0) = \phi(x + (-x)) = \phi(x)\phi(-x),$$

so

$$\phi(-x) = \phi(x)^{-1}.$$

Since also $|\phi(-x)| \leq 1$, it follows that

$$|\phi(x)| = 1 \qquad (x \in G).$$

Thus ϕ is a continuous character of G, i.e. $\phi \in \Gamma$.

Finally, since Rudin defines the Fourier transform by

$$\widehat{f}(\gamma) = \int_G f(x)(-x, \gamma)\, dx,$$

we write the character ϕ in the form

$$\phi(x) = (-x, \gamma)$$

for a unique $\gamma \in \Gamma$. Therefore

$$h(f) = \int_G f(x)\phi(x)\, dx = \int_G f(x)(-x, \gamma)\, dx = \widehat{f}(\gamma).$$

Thus every nonzero complex homomorphism of $L^1(G)$ into $\mathbb{C}$ is obtained from evaluation of the Fourier transform at a unique character $\gamma \in \Gamma$.

To see uniqueness, suppose

$$\widehat{f}(\gamma_1) = \widehat{f}(\gamma_2) \qquad \text{for all } f \in L^1(G).$$

Then

$$\int_G f(x)\big[(-x,\gamma_1)-(-x,\gamma_2)\big]\,dx = 0 \qquad (f \in L^1(G)).$$

Hence

$$(-x,\gamma_1) = (-x,\gamma_2)$$

for almost every $x \in G$. Since both sides are continuous characters, they agree everywhere. Therefore

$$\gamma_1 = \gamma_2.$$

□

Remark. *This theorem identifies the continuous characters of G with the nonzero multiplicative linear functionals on the convolution algebra $L^1(G)$. In other words, the maximal ideal space of $L^1(G)$ is naturally identified with the dual group $\Gamma = \widehat{G}$. This is the point at which Fourier analysis on G begins to merge with the theory of Banach algebras.*

6 The Fourier Transform

Let G be a locally compact abelian group, written additively, and let

$$\Gamma = \widehat{G}$$

be its dual group. Thus Γ consists of all continuous characters

$$\gamma : G \to \mathbb{T}.$$

We denote

$$(x,\gamma) = \gamma(x).$$

Definition 6.1 (Fourier transform)**.** *For $f \in L^1(G)$, the* Fourier transform *of f is the function $\widehat{f}$ on Γ defined by*

$$\widehat{f}(\gamma) = \int_G f(x)(-x,\gamma)\,dx, \qquad \gamma \in \Gamma.$$

By Theorem 5.2, for each fixed $\gamma \in \Gamma$, the map

$$f \longmapsto \widehat{f}(\gamma)$$

is a nonzero complex homomorphism of the Banach algebra $L^1(G)$ into $\mathbb{C}$. Conversely, every nonzero complex homomorphism of $L^1(G)$ arises in this way from a unique character $\gamma \in \Gamma$.

Thus the Fourier transform is precisely the Gelfand transform of the Banach algebra $L^1(G)$, once the maximal ideal space of $L^1(G)$ is identified with the dual group Γ.

We denote by

$$A(\Gamma)$$

the set of all Fourier transforms:

$$A(\Gamma) = \{\widehat{f} : f \in L^1(G)\}.$$

Definition 6.2 (Separating and self-adjoint subalgebras)**.** *Let X be a topological space. A subalgebra $\mathcal{A} \subset C_0(X)$ is called* separating *if for any two distinct points $x, y \in X$, there exists $F \in \mathcal{A}$ such that*

$$F(x) \neq F(y).$$

It is called self-adjoint *if*

$$F \in \mathcal{A} \qquad \Longrightarrow \qquad \overline{F} \in \mathcal{A}.$$

Here $C_0(\Gamma)$ denotes the space of continuous functions on Γ which vanish at infinity.

Theorem 6.3 (Stone–Weierstrass theorem for locally compact spaces)**.** *Let X be a locally compact Hausdorff space, and let*

$$\mathcal{A} \subset C_0(X)$$

be a subalgebra. Suppose that

1. *$\mathcal{A}$ separates points of X, i.e. for every $x \neq y$ in X, there exists $f \in \mathcal{A}$ such that*

$$f(x) \neq f(y);$$

2. *$\mathcal{A}$ is self-adjoint, i.e.*

$$f \in \mathcal{A} \implies \overline{f} \in \mathcal{A};$$

3. *$\mathcal{A}$ vanishes nowhere, i.e. for every $x \in X$, there exists $f \in \mathcal{A}$ such that*

$$f(x) \neq 0.$$

Then $\mathcal{A}$ is uniformly dense in $C_0(X)$. That is, for every $F \in C_0(X)$ and every $\varepsilon > 0$, there exists $f \in \mathcal{A}$ such that

$$\|F - f\|_\infty < \varepsilon.$$

Theorem 6.4 (Basic properties of the Fourier transform)**.** *Let*

$$A(\Gamma) = \{\widehat{f} : f \in L^1(G)\}.$$

Then the following hold.

(a) *$A(\Gamma)$ is a separating self-adjoint subalgebra of $C_0(\Gamma)$. Hence, by the Stone–Weierstrass theorem,*

$$\overline{A(\Gamma)}^{\|\cdot\|_\infty} = C_0(\Gamma).$$

(b) *The Fourier transform of $f * g$ is*

$$\widehat{f * g} = \widehat{f}\,\widehat{g}.$$

(c) *$A(\Gamma)$ is invariant under translations on Γ and under multiplication by characters of Γ of the form*

$$\gamma \mapsto (x, \gamma), \qquad x \in G.$$

(d) *The Fourier transform, viewed as a map*

$$L^1(G) \longrightarrow C_0(\Gamma), \qquad f \mapsto \widehat{f},$$

is norm-decreasing. In particular,

$$\|\widehat{f}\|_\infty \leq \|f\|_1.$$

(e) *If $f \in L^1(G)$ and $\gamma \in \Gamma$, then*

$$(f * \gamma)(x) = (x, \gamma)\widehat{f}(\gamma).$$

Proof. For $f \in L^1(G)$, define

$$f^*(x) = \overline{f(-x)}.$$

Then $f^* \in L^1(G)$, and

$$\widehat{f^*}(\gamma) = \overline{\widehat{f}(\gamma)}.$$

Indeed,

$$\begin{aligned}
\widehat{f^*}(\gamma) &= \int_G \overline{f(-x)}(-x,\gamma)\,dx \\
&= \int_G \overline{f(y)}(y,\gamma)\,dy \\
&= \overline{\int_G f(y)(-y,\gamma)\,dy} \\
&= \overline{\widehat{f}(\gamma)}.
\end{aligned}$$

Thus $A(\Gamma)$ is self-adjoint.

By Theorem 5.2, distinct characters induce distinct homomorphisms on $L^1(G)$. Hence, if $\gamma_1 \neq \gamma_2$, there exists $f \in L^1(G)$ such that

$$\widehat{f}(\gamma_1) \neq \widehat{f}(\gamma_2).$$

Thus $A(\Gamma)$ separates points of Γ. Also, by Theorem 5.2,

$$\widehat{f * g}(\gamma) = \widehat{f}(\gamma)\widehat{g}(\gamma),$$

so $A(\Gamma)$ is closed under multiplication. Therefore $A(\Gamma)$ is a separating self-adjoint subalgebra of $C_0(\Gamma)$. The density statement then follows from the locally compact form of the Stone–Weierstrass Theorem 6.3.

Part (b) is precisely the identity just used:

$$\widehat{f * g} = \widehat{f}\,\widehat{g}.$$

We now prove (c). First let $\gamma_0 \in \Gamma$, and define

$$g(x) = (x,\gamma_0)f(x).$$

Then $g \in L^1(G)$, and

$$\begin{aligned}
\widehat{g}(\gamma) &= \int_G f(x)(x,\gamma_0)(-x,\gamma)\,dx \\
&= \int_G f(x)(-x,\gamma-\gamma_0)\,dx \\
&= \widehat{f}(\gamma-\gamma_0).
\end{aligned}$$

Thus translations of $\widehat{f}$ on Γ again belong to $A(\Gamma)$.

Next, for $x \in G$, define the translate

$$f_x(y) = f(y-x).$$

Then

$$\widehat{f_x}(\gamma) = \int_G f(y-x)(-y,\gamma)\,dy.$$

Putting $u = y - x$, so that $y = u + x$, gives

$$\begin{aligned}
\widehat{f_x}(\gamma) &= \int_G f(u)(-(u+x),\gamma)\,du \\
&= (-x,\gamma)\int_G f(u)(-u,\gamma)\,du \\
&= (-x,\gamma)\widehat{f}(\gamma).
\end{aligned}$$

Thus multiplication by $\gamma \mapsto (-x,\gamma)$, and hence also by $\gamma \mapsto (x,\gamma)$, preserves $A(\Gamma)$.

Part (d) follows immediately from the definition:

$$|\widehat{f}(\gamma)| = \left|\int_G f(x)(-x,\gamma)\,dx\right| \leq \int_G |f(x)|\,dx = \|f\|_1.$$

Taking the supremum over $\gamma \in \Gamma$, we obtain

$$\|\widehat{f}\|_\infty \leq \|f\|_1.$$

Finally, for (e), regard γ as the function

$$x \mapsto (x, \gamma).$$

Then

$$(f * \gamma)(x) = \int_G f(x-y)(y,\gamma)\, dy.$$

Put $u = x - y$, so that $y = x - u$. Then

$$\begin{aligned}(f * \gamma)(x) &= \int_G f(u)(x-u,\gamma)\, du \\ &= (x,\gamma) \int_G f(u)(-u,\gamma)\, du \\ &= (x,\gamma)\widehat{f}(\gamma).\end{aligned}$$

In particular, setting $x = 0$ gives

$$\widehat{f}(\gamma) = (f * \gamma)(0).$$

□

Remark. *This theorem is important because it says that Fourier transforms are abundant: although not every function in $C_0(\Gamma)$ is necessarily the Fourier transform of an L^1-function, every function in $C_0(\Gamma)$ can be uniformly approximated by such Fourier transforms.*

Theorem 6.5. *If G is discrete, then Γ is compact. If G is compact, then Γ is discrete.*

Proof. First suppose that G is discrete. Then the algebra $L^1(G)$ has a unit, namely the point mass at 0,

$$e(0) = 1, \qquad e(x) = 0 \quad (x \neq 0).$$

For every $\gamma \in \Gamma$,

$$\widehat{e}(\gamma) = \sum_{x \in G} e(x)(-x,\gamma) = 1.$$

Thus the constant function 1 belongs to $A(\Gamma)$. Since $A(\Gamma) \subset C_0(\Gamma)$, it follows that the constant function 1 vanishes at infinity on Γ. This is possible only if Γ is compact.

Now suppose that G is compact. Normalize Haar measure so that

$$m(G) = 1.$$

For $\gamma \in \Gamma$, we claim that

$$\int_G (x,\gamma)\, dx = \begin{cases} 1, & \gamma = 0, \\ 0, & \gamma \neq 0. \end{cases}$$

If $\gamma = 0$, this is immediate. If $\gamma \neq 0$, choose $x_0 \in G$ such that

$$(x_0, \gamma) \neq 1.$$

By translation-invariance of Haar measure,

$$\int_G (x,\gamma)\, dx = \int_G (x + x_0, \gamma)\, dx = (x_0,\gamma) \int_G (x,\gamma)\, dx.$$

Since $(x_0, \gamma) \neq 1$, the integral must be 0.

Now take the constant function $1 \in L^1(G)$. Its Fourier transform is

$$\widehat{1}(\gamma) = \int_G (-x, \gamma)\, dx = \begin{cases} 1, & \gamma = 0, \\ 0, & \gamma \neq 0. \end{cases}$$

Thus $\widehat{1}$ is the characteristic function of the singleton $\{0\}$. Since $\widehat{1} \in A(\Gamma) \subset C_0(\Gamma)$, it is continuous. Therefore $\{0\}$ is open in Γ.

Since Γ is a topological group, every translate of the open set $\{0\}$ is open. Hence every singleton in Γ is open, and so Γ is discrete. □

Remark. *The topology on Γ is the Gelfand topology coming from the Banach algebra $L^1(G)$. Equivalently, it is the weakest topology making all functions*

$$\gamma \mapsto \widehat{f}(\gamma), \qquad f \in L^1(G),$$

continuous. We shall now identify this with the usual compact-open topology on the character group.

Equivalently, a net (γ_α) in Γ converges to $\gamma \in \Gamma$ in the Gelfand topology if and only if

$$\widehat{f}(\gamma_\alpha) \longrightarrow \widehat{f}(\gamma) \qquad \text{for every } f \in L^1(G).$$

Theorem 6.6. (a) *The pairing*

$$G \times \Gamma \longrightarrow \mathbb{T}, \qquad (x, \gamma) \longmapsto (x, \gamma),$$

is continuous.

(b) *Let K and C be compact subsets of G and Γ, respectively. Let U_r be the set of all complex numbers z with*

$$|1 - z| < r.$$

Define

$$N(K, r) = \{\gamma \in \Gamma : (x, \gamma) \in U_r \text{ for all } x \in K\},$$

and

$$N(C, r) = \{x \in G : (x, \gamma) \in U_r \text{ for all } \gamma \in C\}.$$

Then $N(K, r)$ and $N(C, r)$ are open subsets of Γ and G, respectively.

(c) *The family of all sets $N(K, r)$ and their translates is a base for the topology of Γ.*

(d) Γ *is a locally compact abelian group.*

Remark. *This theorem is important because it allows us to view $\widehat{G}$ not merely as the maximal ideal space of $L^1(G)$, but as the familiar topological group of continuous characters equipped with the compact-open topology.*

Examples of Dual Groups

Example 6.7 (The dual group of $\mathbb{R}$)**.** *Let*

$$G = \mathbb{R}$$

with addition. For each real number $\xi \in \mathbb{R}$, define

$$\gamma_\xi(x) = e^{i\xi x}.$$

Then γ_ξ is a continuous character of $\mathbb{R}$, and every continuous character of $\mathbb{R}$ is of this form. Thus

$$\widehat{\mathbb{R}} \cong \mathbb{R}.$$

In the pairing notation, we may write

$$(x, \xi) = e^{i\xi x}.$$

Hence, for $f \in L^1(\mathbb{R})$,

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x}\, dx.$$

The topology on $\widehat{\mathbb{R}}$ is originally the Gelfand topology: the weakest topology making all maps

$$\xi \mapsto \widehat{f}(\xi), \qquad f \in L^1(\mathbb{R}),$$

continuous. By Theorem 6.6, this agrees with the compact-open topology on characters. Under the identification

$$\xi \longleftrightarrow \gamma_\xi,$$

this is precisely the usual topology on $\mathbb{R}$.

Example 6.8 (The dual group of $\mathbb{Z}$)**.** *Let*

$$G = \mathbb{Z}$$

with the discrete topology. A character $\gamma \in \widehat{\mathbb{Z}}$ is determined by its value at 1, since

$$\gamma(n) = \gamma(1)^n.$$

Conversely, every $z \in \mathbb{T}$ defines a character by

$$\gamma_z(n) = z^n.$$

Therefore

$$\widehat{\mathbb{Z}} \cong \mathbb{T}.$$

Writing $z = e^{it}$, one has

$$(n, t) = e^{int}.$$

For $f \in \ell^1(\mathbb{Z})$, the Fourier transform becomes

$$\widehat{f}(t) = \sum_{n \in \mathbb{Z}} f(n) e^{-int},$$

which is the usual Fourier series associated to the sequence f.

Example 6.9 (The dual group of $\mathbb{T}$)**.** *Let*

$$G = \mathbb{T}.$$

A character of $\mathbb{T}$ has the form

$$z \mapsto z^n$$

for a unique $n \in \mathbb{Z}$. Therefore

$$\widehat{\mathbb{T}} \cong \mathbb{Z}.$$

Equivalently, writing $z = e^{it}$, the characters are

$$t \mapsto e^{int}, \qquad n \in \mathbb{Z}.$$

Thus for $f \in L^1(\mathbb{T})$, the Fourier transform is the sequence

$$\widehat{f}(n) = \int_{\mathbb{T}} f(z) z^{-n}\, dz,$$

or, under the parametrization $z = e^{it}$ with normalized Haar measure,

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-int}\, dt.$$

These are precisely the classical Fourier coefficients.

Example 6.10 (Finite cyclic groups)**.** *Let*

$$G = \mathbb{Z}/N\mathbb{Z}.$$

For each $k \in \mathbb{Z}/N\mathbb{Z}$, define

$$\gamma_k(n) = e^{2\pi i k n/N}.$$

These are exactly the characters of G, so

$$\widehat{\mathbb{Z}/N\mathbb{Z}} \cong \mathbb{Z}/N\mathbb{Z}.$$

The Fourier transform is the finite Fourier transform:

$$\widehat{f}(k) = \sum_{n=0}^{N-1} f(n) e^{-2\pi i k n/N}.$$

References

[1] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, John Wiley & Sons, 1999.

[2] G. B. Folland, *A Course in Abstract Harmonic Analysis*, CRC Press, 2016.

[3] W. Rudin, *Fourier Analysis on Groups*, Courier Dover Publications, 2017.