# AN ESTIMATE FOR THE NUMBER OF SOLUTIONS OF FORMS IN PRIME VARIABLES

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**Abstract.** In 2021, J. Liu and L. Zhao [6] proved the existence of solutions of a system of $R$ forms with at least $4^{d+2}d^2R^5$ prime variables. Using the Hardy-Littlewood Circle method they found an estimate for the number of such solutions which confirms the existence of prime solutions of a system of forms satisfying some local conditions. In this article, we obtain an unweighted estimate for $R=1$ using a minimalist approach derived by K. Biggs and J. Brandes [1].

## 1. INTRODUCTION

A Diophantine equation is a polynomial equation in one or several variables which only accepts integer solutions. The history of diophantine equations dates back to Diophantus of Alexandria around 250 AD. Since then problems concerning the solution of diophantine equations has been a central theme of research in mathematics. Some famous examples of such problems are the problem of pythagorean triples, Fermat's last Theorem, the Goldbach Conjecture, Waring's problem, etc. Some of these problems have been solved using elementary methods from number theory while others required complicated and advanced tools in number theory and geometry developed recently. A variant of Goldbach problem a.k.a the Ternary Goldbach Conjecture has been solved in 2013 by H.A. Helfgott [5], and significant progress has been made by mathematicians towards the soution of the Waring Problem (c.f. [4]) over decades using tools from Analytic Number Theory. However, a complete solution for either of these problems is currently out of reach. Although Diophantine equations mostly consider solutions in terms of general integers, some variants of the problems mentioned above such as the Ternary Goldbach problem, the Waring-Goldbach problem, the Twin Prime Conjecture, etc. consider solutions in prime numbers.

In recent times, the problem of exploring the existence and the number of prime solutions of forms, which are homogeneous polynomials with integer coefficients, have gained much interest among number theorists. Birch [2] in 1962 gave a method to find integral solutions of forms with sufficiently large number of variables. Much of our work is based on the methods invented by Birch.

We consider a system of equations of homogeneous polynomials of degree $d \geq 2$

$$F_i(\mathbf{x}) = F_i(x_1,\ldots,x_n) \in \mathbb{Z}[x_1,\ldots,x_n],\ 1 \leq i \leq R. \tag{1.1}$$

We are particularly interested in the equation

$$\mathbf{F}(x_1,\ldots,x_n) = 0, \tag{1.2}$$

where $x_1,\ldots,x_n$ are primes. B. Cook and A. Magyar [3] in 2014, were able to establish the solution of (1.2) for large $n$. In fact, they established that that the upper bound for the minimum number of prime solutions of (1.2) $n$ is an exponential tower in $d$. Later in 2021, J. Liu and L. Zhao [6], reduced the upper bound for the minimum number of prime solutions $n$ of (1.2) satisfying some local conditions to $4^{d+2}d^2R^5$. On the other hand, S. Yamagishi [7] using a different method showed that the minimum number of prime solutions $n$ of (1.2) for a single form satisfying some local conditions to be

$2^8 3^4 5^2 d^3 (2d-1)^2 4^d$.

In this article, we will deduce an unweighted version of the result obtained by J. Liu and L. Zhao in [6] for $R = 1$ using a minimalist approach as presented by K. Biggs and J. Brandes in [1]. J. Liu and L. Zhao have already established the existence of prime solutions of (1.2) satisfying some local conditions in [6] Theorem 1.1 . They proved it using the Hardy-Littlewood circle method.

## 2. Formulations and Notations

For the sake of clarity, we follow the notations in [6]. Let $\mathcal{B}$ be a fixed box in the $n$-dimensional space determined by $b'_j < x_j \le b''_j$, $0 < b'_j < b''_j < 1$. Let $P$ be a sufficiently large positive integer and $P\mathcal{B}$ denote the set of all vectors $\mathbf{x} = (x_1, \dots, x_n)$ such that $P^{-1}\mathbf{x} \in \mathcal{B}$. We define

$$N_F(P) = \sum_{\substack{\mathbf{x}\in P\mathcal{B} \\ F(\mathbf{x})=0}} \prod_{i=1}^{n} \mathbb{I}_{\mathbb{P}}(x_i) \tag{2.1}$$

where $F \in \mathbb{Z}[x_1, \dots, x_n]$ is a homogeneous polynomial of degree $d \ge 2$, $\mathbb{P} \subset \mathbb{Z}$ denotes the set of prime numbers, and for $x \in \mathbb{Z}$

$$\mathbb{I}_{\mathbb{P}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{P} \\ 0 & \text{otherwise.} \end{cases}$$

Let $V \subset \mathbb{A}^n$ be the variety defined as $V = V_F = \{\mathbf{x} \in \mathbb{A}^n : F(\mathbf{x}) = 0\}$, and $V_F^* = \{\mathbf{x} \in \mathbb{A}^n : \nabla F(\mathbf{x}) = 0\}$ denote the singular locus of $V$, where $\nabla F(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$.

**Theorem 2.1.** *Let $F \in \mathbb{Z}[x_1, \dots, x_n]$ a homogeneous polynomial of degree $d \ge 2$ satisfying $\nabla F(\mathbf{x}) \ne 0$. Let $N_F(P)$ be defined in (2.1). Suppose, that $n \ge 12 \cdot 2^d (2^d+1)(d-1) + 4d + 8$. For a sufficiently large $A > 0$, we have*

$$N_F(P) = \mathfrak{S}_F^* \mathfrak{I}_F P^{n-d} (\log P)^{-n} + O\left(P^{n-d} (\log P)^{-A}\right), \tag{2.2}$$

*where the singular integral $\mathfrak{I}_F$ and the singular series $\mathfrak{S}_F^*$ are defined in section 5 and section 6, respectively.*

Then, similar to as noted in [6], Theorem 1.1 follows from Theorem 2.1 for $F$. Next, we define some objects and their notations which will be used in the sections to follow.

We use the standard notation $e(z)$ to denote $e^{2i\pi z}$. We also note that for $\mathbf{x} \in \mathbb{Z}^n$, $1 \le \mathbf{x} \le X$ denotes $1 \le x_j \le X$ for all $1 \le j \le n$.

For a fixed positive integer $i$, let the sequence of real numbers $\{\lambda_i(x)\}$ satisfy $\lambda_i(x) = \mathbb{I}_{\mathbb{P}}(x)$ for all $x \ge 1$. Then, for $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{N}^n$, we define $\lambda(\mathbf{x}) = \prod_{i=1}^{n} \lambda_i(x_i)$.

We define the exponential sum

$$S_F(\alpha) = \sum_{1 \le \mathbf{x} \le P} \lambda(\mathbf{x}) e(\alpha.F(\mathbf{x})), \tag{2.3}$$

where $\alpha \in \mathbb{R}$. The major arcs are defined as

$$\mathfrak{M}(Q) = \bigcup_{1 \le q \le Q} \bigcup_{\substack{1 \le a \le Q \\ (a,q)=1}} \left[\frac{a}{q} - \frac{Q}{qP^d}, \frac{a}{q} + \frac{Q}{qP^d}\right]. \tag{2.4}$$

Then for $Q \le \frac{1}{4} P^{\frac{d}{2}}$, the minor arcs are defined as

$$\mathfrak{m}(Q) = \left[P^{-\frac{d}{2}}, 1 + P^{-\frac{d}{2}}\right] \setminus \mathfrak{M}(Q). \tag{2.5}$$

## 3. Mean Value Estimate

As defined in [6], let $h(\mathbf{x},\mathbf{w})$ is a polynomial of $(\mathbf{x},\mathbf{w})$, and the degree of $h$ with regard to $\mathbf{x}$ is smaller than $d$, in other words, $\deg_{\mathbf{x}}(h) < d$. For $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{Z}^k$, we define

$$\mathcal{E}(\alpha;\mathbf{x}) = \sum_{1\le \mathbf{w}\le P} \lambda(\mathbf{w}) e(\alpha\cdot h(\mathbf{x},\mathbf{w})),$$

where $\mathbf{w}\in\mathbb{N}^t$. Similarly, we define

$$\mathcal{T}(\alpha;\mathbf{x}) = \sum_{1\le \mathbf{u}\le P} \lambda(\mathbf{u}) e(\alpha\cdot H(\mathbf{x},\mathbf{u})),$$

where $\mathbf{u}\in\mathbb{N}^l$, $H(\mathbf{x},\mathbf{u})$ is a polynomial of $(\mathbf{x},\mathbf{u})$, and the degree of $H$ with regard to $\mathbf{x}$ is smaller than $d$. When $t=0$, we consider $\mathcal{E}(\alpha;\mathbf{x})\equiv 1$. Let $g$ be a form of degree $d$ in $\mathbf{x}$, and $\mathfrak{n}\subset[0,2]$ be a measurable set. Then, we define

$$\mathcal{J} := \mathcal{J}_{\mathfrak{n}} = \sum_{1\le\mathbf{x}\le P}\left|\int_{\mathfrak{n}} e(\alpha\cdot g(\mathbf{x}))\mathcal{E}(\alpha;\mathbf{x})\mathcal{T}(\alpha;\mathbf{x})\,\mathrm{d}\alpha\right|^{\mathbf{2}}. \tag{3.1}$$

Using $\lambda_i(x) = \mathbb{I}_{\mathbb{P}}(x)$ instead of $|\lambda_i(x)|\le \log x$ for all $i\in\{1,\dots,n\}$ in the proof of Lemma 4.1 in [6], we obtain the following estimate for $\mathcal{J}$.

**Lemma 3.1.** *Let $\mathcal{J}$ be as defined in (3.1). Let $\kappa>0$ be a real number satisfying $\kappa > 2^d(d-1)$. Suppose that $k-\dim V_g^*\ge\kappa$. Let $X\le \frac{1}{4}P^{\frac{d}{2}}$. Then, we have*

$$\begin{aligned}\mathcal{J} \ll\ & P^{k+2t+2l-d}(\log P)^k X^{2-\frac{\kappa}{2^{d-1}(d-1)}}|\mathfrak{n}| \\ &+ X^2P^{-d}\sum_{1\le\mathbf{x}\le P}\int_{\mathfrak{n}}|\mathcal{E}(\alpha;\mathbf{x})\mathcal{T}(\alpha;\mathbf{x})|^2\,\mathrm{d}\alpha,\end{aligned}$$

*where the implied constant depends only on (the coefficients of) $g$.*

Following [6], let $G(\mathbf{x},\mathbf{y})$ be a polynomial in $(\mathbf{x},\mathbf{y})$, and the degree of $G$ as a polynomial of $\mathbf{x}$ is smaller than $d$. For $x\in\mathbb{Z}^m$, we define

$$\mathcal{T}_0(\alpha;\mathbf{x}) = \sum_{1\le\mathbf{y}\le P}\lambda(y)e(\alpha\cdot G(\mathbf{x},\mathbf{y})),$$

where $y\in\mathbb{N}^l$. Let

$$I := I_{\mathfrak{n}} = \sum_{1\le\mathbf{x}\le P}\left|\int_{\mathfrak{n}} e(\alpha\cdot f(\mathbf{x}))\mathcal{T}_0(\alpha;\mathbf{x})\,\mathrm{d}\alpha\right|^2, \tag{3.2}$$

where $f$ is a form of degree $d$. Putting $k=m$, $l=0$, and $\mathcal{T}(\alpha;\mathbf{x}) = \mathcal{T}(\alpha;\mathbf{x})$ in Lemma 3.1, we obtain the following version of Lemma 4.2 in [6].

**Lemma 3.2.** *Let $I$ be as defined in (3.2). Let $\kappa>0$ be a real number satisfying $\kappa>2^d(d-1)$. Suppose that $m-\dim V_g^*\ge\kappa$. Let $X\le\frac{1}{4}P^{\frac{d}{2}}$. Then, we have*

$$\begin{aligned}I \ll\ & P^{m+2l-d}(\log P)^m X^{2-\frac{\kappa}{2^{d-1}(d-1)}}|\mathfrak{n}| \\ &+ X^2P^{-d}\sum_{1\le\mathbf{x}\le P}\int_{\mathfrak{n}}|\mathcal{T}_0(\alpha;\mathbf{x})|^2\,\mathrm{d}\alpha.\end{aligned}$$

Next, for a form of $F$ of degree $d$ in $n$ variables, we write $\mathbf{x}=(\mathbf{y},\mathbf{z},\mathbf{w})$, where $\mathbf{y}\in\mathbb{N}^m, \mathbf{z}\in\mathbb{N}^s$, $\mathbf{w}\in\mathbb{N}^t$, and $m+s+t=n$. Then $F$ can be uniquely written as,

$$F = f(\mathbf{y}) + g(\mathbf{y},\mathbf{z}) + h(\mathbf{y},\mathbf{z},\mathbf{w}) \tag{3.3}$$

where the degree of $g$ as a polynomial of $\mathbf{y}$ is smaller than $d$ and the degree of $h$ as a polynomial of $(\mathbf{y},\mathbf{z})$ is also smaller than $d$. Let us write

$$S_F(\alpha) = \sum_{1\le\mathbf{y}\le P}\sum_{1\le\mathbf{z}\le P}\sum_{1\le\mathbf{w}\le P}\lambda(\mathbf{y})\lambda(\mathbf{z})\lambda(\mathbf{w})e(\alpha.F(\mathbf{y},\mathbf{z},\mathbf{w})). \tag{3.4}$$

We define

$$\mathcal{E}_{\mathbf{y},\mathbf{z}}(\alpha) = \sum_{1\le \mathbf{w}\le P} \lambda(\mathbf{w}) e(\alpha.F(\mathbf{y},\mathbf{z},\mathbf{w})). \tag{3.5}$$

Using Lemma 3.1 and Lemma 3.2 in the proof of [6], we obtain the following version of Proposition 5.1 in [6].

**Proposition 3.3.** *Let $S_F(\alpha)$ be given in (3.4). Let $F$ be as described above. Let $X \ll \frac{1}{4}P^{\frac{d}{2}}$. Let $\kappa_1$ and $\kappa_2$ be two real numbers satisfying* $\min(\kappa_1,\kappa_2) > 2^d(d-1)$. *Suppose that $m - \dim V_f^* \ge \kappa_1$ and $m+s-\dim V_g^* \ge \kappa_2$. Then we have*

$$\int_{\mathfrak{n}} S_F(\alpha)\,\mathrm{d}\alpha \ll P^{n-\frac{d}{2}}(\log P)^n X^{1-\frac{\kappa_1}{2^d(d-1)}}|\mathfrak{n}|^{\frac{1}{2}} + P^{n-\frac{3d}{4}}(\log P)^n X^{\frac{3}{2}-\frac{\kappa_2}{2^{d+1}(d-1)}}|\mathfrak{n}|^{\frac{1}{4}} + P^{m+s-d}X^2 \sup(\mathcal{E}),$$

*where* $\sup(\mathcal{E}) = \sup_{\alpha\in\mathfrak{n}} \sup_{\mathbf{y}} \sup_{\mathbf{z}} |\mathcal{E}_{\mathbf{y},\mathbf{z}}(\alpha)|$.

## 4. A Non-Trivial Estimate of $\sup(\mathcal{E})$ and The Minor Arc Estimate

Let $m \ge 1$, $d_1,\dots,d_m \ge 1$, and $d_1+\cdots+d_m = d$. Let $\mathcal{B}_m(P)$ be the box in $m$-dimensional space defined by $b'_jP < x_j \le b''_jP$ where $0 < b'_j < b''_j < 1$ are fixed constants for $1 \le j \le m$. We define

$$\gamma_m(\alpha) = \sum_{x_1,\dots,x_m\in\mathcal{B}_m(P)} \prod_{i=1}^{m} \mathbb{I}_{\mathbb{P}}(x_i) e(f(x_1,\dots,x_m)),$$

where $f$ is a polynomial in $x_1,\dots,x_m$ of degree $d$ with real coefficients. We can assme, $f(x_1,\dots,x_m) = \alpha x_1^{d_1}\dots x_m^{d_m} + g(x_1,\dots,x_m)$, where $g$ is a polynomial with real coefficients, such that the coefficient of $x_1^{d_1}\dots x_m^{d_m}$ in $g$ is zero, $\deg_{x_i}(g) \le d_i$ for all $i \in \{1,\dots,m\}$, and $\deg(g) \le d$.

Then, using $\lambda_i(x) = \mathbb{I}_{\mathbb{P}}(x)$ instead of $|\lambda_i(x)| \le \log x$ for all $i \in \{1,\dots,m\}$ in the Lemma 6.1 in [6], we obtain:

**Lemma 4.1.** *Let $m \ge 2$. Suppose that $Q \le P$, and let $\alpha \in \mathfrak{m}(Q)$. Then we have*

$$\gamma_m(\alpha) \ll P^m (\log P) Q^{-\frac{1}{2^d}}.$$

**Lemma 4.2.** *Let $0 < b' < b'' < 1$ be two fixed constants. Suppose that $Q \le P^{\frac{1}{4}}$. Let $\alpha \in \mathfrak{m}(Q)$. Then we have*

$$\sum_{b'P\le x\le b''P} \mathbb{I}_{\mathbb{P}}(x) e(f(x)) \ll P(\log P)^6 Q^{-\frac{1}{2^d}+\varepsilon},$$

*for arbitrary $\varepsilon > 0$.*

*Proof.* For primes $p \in \mathbb{P}$,

$$\begin{aligned}\sum_{b'P<x\le b''P} \mathbb{I}_{\mathbb{P}}(x) e(f(x)) &= \sum_{b'P<p\le b''P} e(f(p)) \\ &= \sum_{b'P<x\le b''P} \frac{\log x}{\log x} e(f(x)) \mathbb{I}_{\mathbb{P}}(x)\end{aligned}$$

Then, using partial summation we obtain

$$\begin{aligned}\left|\sum_{b'P<x\le b''P}\frac{\log x}{\log x}e(f(x))\mathbb{I}_{\mathbb{P}}(x)\right| &\le \frac{1}{\log P}\left|\sum_{b'P<x\le b''P}e(f(x))\mathbb{I}_{\mathbb{P}}(x)\log x\right| \\ &\quad+\int_{b'P}^{b''P}\left|\sum_{b'P<x\le X}\frac{e(f(x))\mathbb{I}_{\mathbb{P}}(x)\log x}{X(\log X)^2}\right|\mathrm{d}X \\ &\le \frac{1}{\log P}\left|\sum_{b'P<x\le b''P}e(f(x))\mathbb{I}_{\mathbb{P}}(x)\log x\right| \\ &\quad+\left(\sup_{b'P<X\le b''P}\left|\sum_{b'P<x\le X}e(f(x))\mathbb{I}_{\mathbb{P}}(x)\log x\right|\right)\int_{b'P}^{b''P}\frac{1}{|X(\log X)^2|}\mathrm{d}X \\ &\ll \frac{1}{\log P}\sup_{b'P<X\le b''P}\left|\sum_{b'P<x\le X}e(f(x))\mathbb{I}_{\mathbb{P}}(x)\log x\right|\end{aligned}$$

Using Lemma 6.8 in [6], we obtain

$$\sum_{b'P<x\le b''P}\frac{\log x}{\log x}e(f(x))\mathbb{I}_{\mathbb{P}}(x)\ll P(\log P)^6Q^{-\frac{1}{2^d}+\varepsilon}$$

for an arbitrary $\varepsilon>0$. □

Let $h(x_1,\dots,x_t)$ be a form of degree $d$ and let $g(x_1,\dots,x_t)$ be a polynomial of $\mathbf{x}=(x_1,\dots,x_t)$ of degree smaller than $d$ such that

$$f=g+h.$$

We define

$$\mathcal{E}(\alpha)=\sum_{x_1,\dots,x_t\in\mathcal{B}_t(P)}\prod_{i=1}^{t}\mathbb{I}_{\mathbb{P}}(x_i)e(\alpha f(x_1,\dots,x_m)). \tag{4.1}$$

Taking $R=1$, and using Lemma 4.2 instead of Lemma 6.8 of [6] in the proof of Lemma 6.9 in [6], we obtain the following:

**Lemma 4.3.** *Let $\mathcal{E}(\alpha)$ be given in (4.1). Let h be non-zero. Let $Q\le P^{\frac{1}{4}}$ and $\alpha\in\mathfrak{m}(Q)$. Then we have*

$$\mathcal{E}(\alpha)\ll P^t(\log P)^{t+5}Q^{-\frac{1}{2^d}+\varepsilon}$$

*for arbitrary $\varepsilon>0$.*

Using Proposition 3.3 and Lemma 4.3 we obtain a non-trivial bound of $\sup(\mathcal{E})$ which follow as a result of the following lemma analogous to Lemma 6.10 in [6] for $R=1$.

**Lemma 4.4.** *Let $\mathcal{E}(\alpha)$ be given in (4.1). Let h be a non-zero polynomial. Let $Q\le\frac{1}{4}P^{\frac{d}{2}}$ and $\alpha\in\mathfrak{m}(Q)$. Then we have*

$$\mathcal{E}(\alpha)\ll P^t(\log P)^{t+5}Q^{-\frac{1}{2^{d+1}d}+\varepsilon}$$

*for arbitrary $\varepsilon>0$.*

Let $F$ be a form of degree $d$ as represented uniquely in (3.3). Then, we can represent $h$ uniquely as

$$h(\mathbf{y},\mathbf{z},\mathbf{w})=G(\mathbf{y},\mathbf{z},\mathbf{w})+H(\mathbf{w}), \tag{4.2}$$

where $G$ is a polynomial with $\deg_{\mathbf{w}}(G)<d$, $H$ is a form in $\mathbf{w}\in\mathbb{N}^t$, $\mathbf{y}\in\mathbb{N}^m$, and $\mathbf{z}\in\mathbb{N}^s$. Then using Proposition 3.3 and Lemma 4.4 instead of Proposition 5.1 and Lemma 6.10 of [6] in the proof of Lemma 7.1 in [6], we obtain the following analogue of Lemma 7.1 of [6] for $R=1$.

**Lemma 4.5.** *Let $F$ be a form of degree $d$ in $n$ variables with the unique representation as in (3.3) and (4.2). Let $L < \frac{1}{4}P^{\frac{d}{2}}$. Let $\mathfrak{m}(L)$ be as given in (2.5). Suppose that*

(i) $m - \dim V_f^* \geq \kappa_1 = (2^{d+2}+1)\cdot 2^d(d-1)+1$,
(ii) $m+s-\dim V_g^* \geq \kappa_2 = (2^{d+2}+3)\cdot 2^d(d-1)+1$, *and*
(iii) *$H$ is non-zero.*

*Then there exists a positive number $\delta > 0$ (depending on $d$) such that*

$$\int_{\mathfrak{m}(L)} S_F(\alpha)\,\mathrm{d}\alpha \ll P^{n-d}(\log P)^{n+5}L^{-\delta}.$$

As shown in [6], we apply Lemma 8.1 and Lemma 8.2 of [6] for $R=1$, to obtain a minor arc estimate without the conditions mentioned in Lemma 4.5.

**Lemma 4.6.** *Let $F$ be a form of degree $d$ in $n$ variables and $\nabla F \neq 0$. Let $L < \frac{1}{4}P^{\frac{d}{2}}$. Let $\mathfrak{m}(L)$ be as defined (2.5) and $\alpha \in \mathfrak{m}(L)$. Suppose $n \geq 2\kappa_2+\kappa_1+4d+5 = 2^d(3\cdot 2^{d+2}+7)(d-1)+4d+8$. Then there exists a positive number $\delta>0$ (depending on $d$) such that*

$$\int_{\mathfrak{m}(L)} S_F(\alpha)\,\mathrm{d}\alpha \ll P^{n-d}(\log P)^{n+5}L^{-\delta}.$$

## 5. THE SINGULAR INTEGRAL

Recall the $n$-dimensional box $\mathcal{B}$ determined by $b_j' < x_j \leq b_j''$, $0 < b_j' < b_j'' < 1$, $1 \leq j \leq n$. We define

$$\mathcal{I}_P(\beta) = \int_{Pb_1'}^{Pb_1''}\cdots\int_{Pb_n'}^{Pb_n''} \frac{e(\beta F(x_1,\ldots,x_n))}{\prod_{i=1}^n \log x_i}\,\mathrm{d}x_1\ldots\mathrm{d}x_n, \tag{5.1}$$

and

$$\nu(\beta) = \int_{\mathcal{B}} e(\beta F(\mathbf{x}))\,\mathrm{d}\mathbf{x}. \tag{5.2}$$

Let $\mathrm{li} : [2,+\infty) \to [0,+\infty)$ be defined as

$$\mathrm{li}(x) := \int_2^x \frac{\mathrm{d}t}{\log t}.$$

Now, we refer to [1] to determine a change of variables to express $\mathcal{I}_P(\beta)$ in terms of $\nu(\beta)$. Following section 4 of [1], we note that we can rewrite (5.1) as

$$\mathcal{I}_P(\beta) = (\mathrm{li}(P))^n \int_{\mathcal{B}} e(\beta F(\mathbf{li}^{-1}(\mathrm{li}(P)\mathbf{x}))\,\mathrm{d}\mathbf{x}, \tag{5.3}$$

where $\mathbf{li}^{-1}(\mathbf{x}) := (\mathrm{li}^{-1}(x_1),\ldots,\mathrm{li}^{-1}(x_n))$.

Moreover, we define $\theta = \beta P^d$, $\mathbf{z} := \mathbf{z}(\mathbf{x}) = (z_1(x_1),\ldots,z_n(x_n))$, where $z_i = z_i(x_i) := P^{-1}\,\mathrm{li}^{-1}(\mathrm{li}(P)x_i)$, $1 \leq i \leq n$ and

$$\omega_{\mathbb{P}}(\theta) = \int_{\mathcal{B}} e(\theta F(\mathbf{z}(\mathbf{x})))\,\mathrm{d}\mathbf{x}. \tag{5.4}$$

Let $\log(\mathbf{z}) := (\log z_1,\ldots,\log z_n)$. Then as mentioned in sections 5 and 7 of [1], we define the $\mathbb{P}$-normalized measure of level $P$ on $\mathbb{R}^+$ as

$$\mathrm{d}\sigma_{\mathbb{P}}(z_i) := \frac{P}{\mathrm{li}(P)}\frac{1}{\log(Pz_i)}\mathrm{d}z_i, \tag{5.5}$$

and, we denote

$$\mathrm{d}\sigma_{\mathbb{P}}(\mathbf{z}) := \prod_{i=1}^n \mathrm{d}\sigma_{\mathbb{P}}(z_i).$$

Then, we can rewrite (5.4) as

$$\omega_{\mathbb{P}}(\theta)=\int_{\mathcal{B}} e(\theta F(\mathbf{z}))\,\mathrm{d}\sigma_{\mathbb{P}}(\mathbf{z})=\left(\frac{\log P}{P}\right)^{n}\left[\int_{\left[Pb_1',Pb_1''\right]}\cdots\int_{\left[Pb_n',Pb_n''\right]}\frac{e(\beta F(z_1,\ldots,z_n))}{\prod_{i=1}^{n}\log(z_i)}\prod_{i=1}^{n}\mathrm{d}z_i\right] \tag{5.6}$$

with some abuse of notation, after the change of variables we relabel the integration variable as $z$.

In the next lemma, we get rid of the logs in the representation of $\omega_{\mathbb{P}}(\theta)$ in (5.6) as follows:

**Lemma 5.1.** *Let $\omega_{\mathbb{P}}(\theta)$ be as defined in (5.6). Then, we obtain:*

$$\omega_{\mathbb{P}}(\theta)=\left(\frac{\log P}{P}\right)^{n}\left[\int_{\left[Pb_1',Pb_1''\right]}\cdots\int_{\left[Pb_n',Pb_n''\right]}\frac{e(\beta F(z_1,\ldots,z_n))}{\prod_{i=1}^{n}\log(z_i)}\prod_{i=1}^{n}\mathrm{d}z_i\right]\sim\frac{1}{P^n}\int_{P\mathcal{B}} e(\beta F(\mathbf{z}))\,\mathrm{d}\mathbf{z}$$

*Proof.* We consider the difference

$$\int_{Pb_1'}^{Pb_1''}\cdots\int_{Pb_n'}^{Pb_n''}\frac{e(\beta F(z_1,\ldots,z_n))}{\prod_{i=1}^{n}\log z_i}\,dz_1\cdots dz_n-\frac{1}{(\log P)^n}\int_{P\mathcal{B}} e(\beta F(z))\,dz, \tag{5.7}$$

and estimate its absolute value.

By the triangle inequality and the bound $|e(\beta F(z))|=1$, we obtain

$$\left|\int_{P\mathcal{B}} e(\beta F(z))\left(\prod_{i=1}^{n}\frac{1}{\log z_i}-\frac{1}{(\log P)^n}\right)dz\right|\le\int_{P\mathcal{B}}\left|\prod_{i=1}^{n}\frac{1}{\log z_i}-\frac{1}{(\log P)^n}\right|dz. \tag{5.7'}$$

We now bound the integrand uniformly on the box $P\mathcal{B}$. For each $i\in\{1,\ldots,n\}$ and each $z\in P\mathcal{B}$, we have

$$Pb_i'\le z_i\le Pb_i'',$$

and hence

$$\log z_i=\log P+\log(z_i/P).$$

Since $z_i/P\in[b_i',b_i'']\subset(0,1)$ is bounded away from 0 and $\infty$, there exists a constant $C_0>0$, depending only on $\mathcal{B}$, such that

$$|\log(z_i/P)|\le C_0\qquad(z\in P\mathcal{B},\ 1\le i\le n). \tag{5.7a}$$

For $P$ sufficiently large we have $\log P\ge 2C_0$, and therefore

$$\left|\frac{1}{\log z_i}-\frac{1}{\log P}\right|=\left|\frac{\log P-\log z_i}{(\log z_i)(\log P)}\right|=\frac{|\log(z_i/P)|}{(\log z_i)(\log P)}\ll\frac{1}{(\log P)^2}, \tag{5.7b}$$

uniformly for $z\in P\mathcal{B}$ and $1\le i\le n$. Consequently,

$$\frac{1}{\log z_i}=\frac{1}{\log P}+O\!\left(\frac{1}{(\log P)^2}\right)\qquad(z\in P\mathcal{B}),$$

with an implied constant depending only on $\mathcal{B}$.

Multiplying (5.7$c$) over $i=1,\ldots,n$ (with $n$ fixed), we obtain

$$\prod_{i=1}^{n}\frac{1}{\log z_i}=\left(\frac{1}{\log P}\right)^{n}+O\!\left(\frac{1}{(\log P)^{n+1}}\right)\qquad(z\in P\mathcal{B}), \tag{5.7d}$$

again uniformly on $P\mathcal{B}$. Hence

$$\left|\prod_{i=1}^{n}\frac{1}{\log z_i}-\frac{1}{(\log P)^n}\right|\ll\frac{1}{(\log P)^{n+1}}\qquad(z\in P\mathcal{B}). \tag{5.7e}$$

Inserting (5.7$e$) into (5.7$'$), we deduce that

$$\int_{P\mathcal{B}}\left|\prod_{i=1}^{n}\frac{1}{\log z_i}-\frac{1}{(\log P)^n}\right|dz\ll\frac{1}{(\log P)^{n+1}}\,\mathrm{vol}(P\mathcal{B}). \tag{5.7f}$$

Since $\mathrm{vol}(P\mathcal{B}) = P^n\,\mathrm{vol}(\mathcal{B})$, this yields

$$\int_{P\mathcal{B}} \left| \prod_{i=1}^{n} \frac{1}{\log z_i} - \frac{1}{(\log P)^n} \right| dz \ll \frac{P^n}{(\log P)^{n+1}}. \tag{5.7g}$$

Multiplying both sides of (5.7$g$) by $\left(\frac{\log P}{P}\right)^n$ and letting $P \to \infty$, we conclude that

$$\left(\frac{\log P}{P}\right)^n \int_{Pb_1'}^{Pb_1''} \cdots \int_{Pb_n'}^{Pb_n''} \frac{e(\beta F(z_1,\cdots,z_n))}{\prod_{i=1}^n \log z_i} \prod_{i=1}^{n} \mathrm{d}z_i - \frac{1}{P^n} \int_{P\mathcal{B}} e(\beta F(z))\, dz \longrightarrow 0.$$

This proves the claim of Lemma 5.1. □

From Lemma 5.1, we obtain

$$\omega_{\mathbb{P}}(\theta) \approx \frac{1}{P^n} \int_{P\mathcal{B}} e(\beta F(\mathbf{z}))\, \mathrm{d}\mathbf{z}. \tag{5.8}$$

From (5.3) and (5.4),we obtain

$$\mathfrak{I}_P(\beta) = (\mathrm{li}(P))^n \omega_{\mathbb{P}}(\theta). \tag{5.9}$$

Substituting (5.8) in (5.9),

$$\mathfrak{I}_P(\beta) = \frac{1}{(\log P)^n} \int_{P\mathcal{B}} e(\beta.F(\mathbf{z}))\, \mathrm{d}\mathbf{z}. \tag{5.10}$$

We further obtain,

$$\mathfrak{I}_P(\beta) = \frac{P^n}{(\log P)^n} \int_{\mathcal{B}} e(P^d \beta F(\mathbf{z}))\, \mathrm{d}\mathbf{z} \tag{5.11}$$

$$= \left(\frac{P}{\log P}\right)^n \nu(P^d \beta). \tag{5.12}$$

By Lemma 5.2 of Birch [2], we have $\nu(\beta) \ll \frac{1}{(|\beta|+1)^3}$. Then the singular integral could be defined as

$$\mathfrak{I}_F = \lim_{Q \to +\infty} \int_{|\beta| \le Q} \nu(\beta)\, \mathrm{d}\beta. \tag{5.13}$$

Further, Lemma 5.3 of Birch [2] gives us the following estimate:

**Lemma 5.2.** *Let $\mathfrak{I}_F$ be as defined in (5.13). Suppose (5.1) holds. Then, we obtain*

$$\left| \mathfrak{I}_F - \int_{|\beta| \le Q} \nu(\beta)\, \mathrm{d}\beta \right| \ll Q^{-\frac{1}{2}}. \tag{5.14}$$

## 6. The Major Arc Estimate

We define

$$S^*(q,a) = S_F^*(q,a) = \sum_{\substack{1 \le \mathbf{b} \le q \\ (\mathbf{b},q)=1}} e\left(\frac{aF(\mathbf{b})}{q}\right), \tag{6.1}$$

where $\mathbf{b} = (b_1, \ldots, b_n)$, and $\gcd(\mathbf{b}, q) = 1$ implies $\gcd(b_i, q) = 1$ for all $i \in \{1, \ldots, n\}$. Let

$$\mathcal{A}^*(q) = \sum_{\substack{1 \le a \le q \\ (a,q)=1}} S^*(a,q).$$

Then, using Lemma 10 in [3], one obtains

$$\frac{1}{\phi(q)^n} \mathcal{A}^*(q) \ll q^{1 - \frac{n - \dim V_F^*}{(d-1)2^d} + \varepsilon}. \tag{6.2}$$

From Lemma 8.2 in [6], we obtain

$$n - \dim V_F^* > 2^{d+1} d. \tag{6.3}$$

Using (6.2) and (6.3), we obtain

$$\frac{1}{\phi(q)^n} \mathcal{A}^*(q) \ll q^{-1-\delta} \tag{6.4}$$

for some $\delta > 0$ (depending on $d$). We define the singular series subject to condition (6.3), as

$$\mathfrak{S}_F^* = \sum_{q=1}^{\infty} \frac{1}{\phi(q)^n} \mathcal{A}^*(q). \tag{6.5}$$

As mentioned in section 3 of [6], to study the major arcs for diophantine equations in prime variables, we define $\mathcal{L} = (\log P)^{A_0}$, where $A_0 > 0$ is sufficiently large. Let the major arc be defined as $\mathfrak{M} = \mathfrak{M}(\mathcal{L})$ and the minor arc as $\mathfrak{m} = \mathfrak{m}(\mathcal{L})$.

**Lemma 6.1.** *Suppose that (6.3) holds and $\delta$ is as defined in (6.4). Then, we have*

$$\int_{\mathfrak{M}} S_F(\alpha)\,\mathrm{d}\alpha = \mathfrak{S}_F^* \mathfrak{I}_F P^{n-d} (\log P)^{-n} + O\left( \frac{P^{n-d}}{(\log P)^n} \mathcal{L}^{-\delta} \right).$$

*Proof.* Using the definition of the major arcs $\mathfrak{M}$, we obtain

$$\int_{\mathfrak{M}} S_F(\alpha)\,\mathrm{d}\alpha = \sum_{q \le \mathcal{L}} \sum_{\substack{1 \le a \le q \\ (a,q)=1}} \int_{|\beta| \le \frac{\mathcal{L}}{qP^d}} S_F\left( \frac{a}{q} + \beta \right) \mathrm{d}\beta. \tag{6.6}$$

Applying the Siegel-Walfisz Theorem and the partial summation formula for $q \le \mathcal{L}$ and $|\beta| \le \frac{\mathcal{L}}{qP^d}$, one can deduce

$$S_F(\frac{a}{q} + \beta) = \frac{1}{\phi(q)^n} S(q,a)^* \mathfrak{I}_P(\beta) + O(P^n \mathcal{L}^{-2023}), \tag{6.7}$$

where the Gauss sum $S(q,a)^*$ is given in (6.1) and the integral $\mathfrak{I}_P(\beta)$ is given in (5.12).

Then, putting (6.7) in (6.6) and using the definition of $\mathcal{A}^*(q)$, we obtain

$$\int_{\mathfrak{M}} S_F(\alpha)\,\mathrm{d}\alpha = \left( \frac{P}{\log P} \right)^n \sum_{q \le \mathcal{L}} \mathcal{A}^*(q) \int_{|\beta| \le \frac{\mathcal{L}}{qP^d}} \nu(P^d \beta)\,\mathrm{d}\beta + O\left( P^{n-d} \mathcal{L}^{-2022} \right). \tag{6.8}$$

With a simple change of variables, we obtain

$$\int_{\mathfrak{M}} S_F(\alpha)\,\mathrm{d}\alpha = \frac{P^{n-d}}{(\log P)^n} \sum_{q \le \mathcal{L}} \mathcal{A}^*(q) \int_{|\beta| \le \frac{\mathcal{L}}{q}} \nu(\beta)\,\mathrm{d}\beta + O\left( P^{n-d} \mathcal{L}^{-2022} \right). \tag{6.9}$$

Applying Lemma 5.2 to (6.9), one obtains

$$\int_{\mathfrak{M}} S_F(\alpha)\,\mathrm{d}\alpha = \frac{P^{n-d}}{(\log P)^n} \sum_{q \le \mathcal{L}} \mathcal{A}^*(q) \mathfrak{I}_F + \frac{P^{n-d}}{(\log P)^n} \sum_{q \le \mathcal{L}} \frac{O\left( (q\mathcal{L}^{-1})^{\frac{1}{2}} \right)}{\phi(q)^n} |\mathcal{A}^*(q)|. \tag{6.10}$$

Using (6.4) in (6.10), we obtain

$$\int_{\mathfrak{M}} S_F(\alpha)\,\mathrm{d}\alpha = \frac{P^{n-d}}{(\log P)^n} \sum_{q \le \mathcal{L}} \mathcal{A}^*(q) \mathfrak{I}_F + O\left( \frac{P^{n-d}}{(\log P)^n} \mathcal{L}^{-\delta} \right). \tag{6.11}$$

Applying (6.4) again, we obtain

$$\sum_{q \le \mathcal{L}} \mathcal{A}^*(q) = \sum_{q=1}^{\infty} \mathcal{A}^*(q) + O(\mathcal{L}^{-\delta}). \tag{6.12}$$

Using (6.5) and (6.12) in (6.11), we obtain the desired result. □

We know, that

$$N_F(P) = \int_0^1 S_F(\alpha)\,\mathrm{d}\alpha = \int_{\mathfrak{M}(\mathcal{L})} S_F(\alpha)\,\mathrm{d}\alpha + \int_{\mathfrak{m}(\mathcal{L})} S_F(\alpha)\,\mathrm{d}\alpha.$$

Then, using Lemma 4.6 and Lemma 6.1, we obtain

$$N_F(P) = \mathfrak{S}_F^* \mathfrak{I}_F P^{n-d} (\log P)^{-n} + O(P^{n-d} (\log P)^{n+5} \mathcal{L}^{-\delta}). \tag{6.13}$$

Choosing $A_0$ sufficiently large, proves Theorem 2.1.

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